

Pre-Master Class

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Master of Finance

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Outline

- 1 Time Value of Money
- 2 Valuing Bonds and Stocks
- 3 Computing Returns and Return Indices
- 4 Descriptive Statistics and Frequency Distributions
- 5 Theoretical Distributions and Hypothesis Testing
- 6 Portfolio Returns and Risk
- 7 Portfolio Selection
- 8 Univariate Regression
- 9 CAPM and the Single Index Model
- 10 Forwards and Futures
- 11 Introduction to Options



Learning Objectives

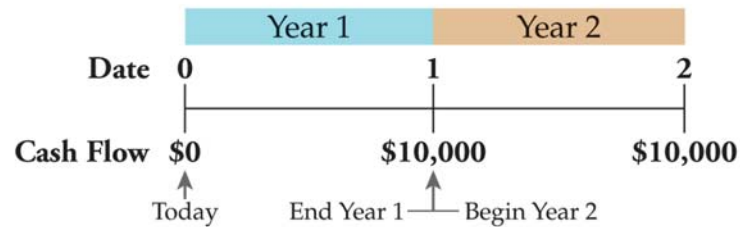
- Draw a time line illustrating a set of cash flows
- Know the different business and day count conventions
- Understand the rules of simple interest rate calculations
- Understand the rules of compounded interest
- Understand the NPV rule

The Time Line

A timeline is a linear representation of the timing of potential cash flows.

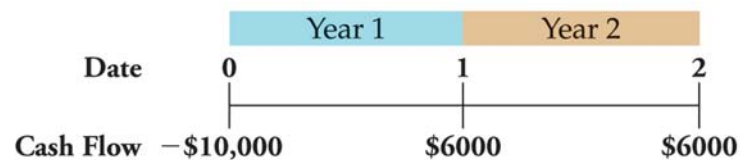
The Time Line

- Inflows are positive cash flows.
- Example: Assume that you made a loan to a friend. You will be repaid in two installments, one at the end of each year over the next two years.



The Time Line

- Outflows are negative cash flows.
- Example: Assume that you are lending \$ 10 000 today and that the loan will be repaid in two annual \$ 6 000 payments.



The Time Line

- Timelines can represent cash flows that take place at the end of any time period – a month, a week, a day, etc.
- *Berk & Demarzo* Textbook Example 4.1

Constructing a Timeline

Problem

Suppose you must pay tuition of \$10,000 per year for the next two years. Your tuition payments must be made in equal installments at the start of each semester. What is the timeline of your tuition payments?

The Time Line

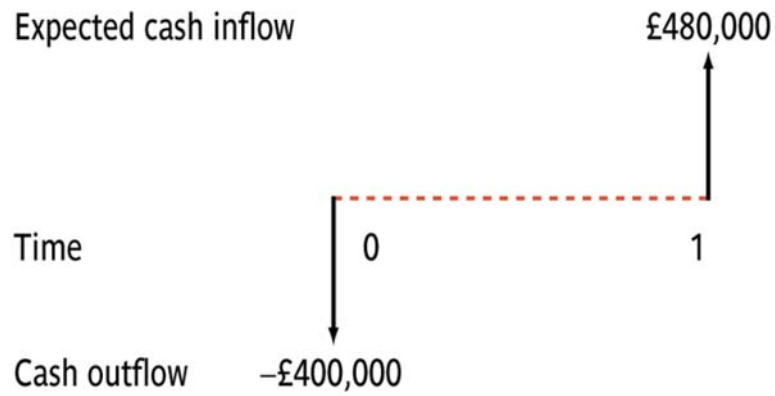
- Timelines can represent cash flows that take place at the end of any time period – a month, a week, a day, etc.
- *Berk & Demarzo* Textbook Example 4.1

Solution

Assuming today is the start of the first semester, your first payment occurs at date 0 (today). The remaining payments occur at semester intervals. Using one semester as the period length, we can construct a timeline as follows:

Date (Semesters)	0	1	2	3	4
Cash Flow	-\$5000	-\$5000	-\$5000	-\$5000	\$0

Alternative Representation of the Cash Flows



Four Basic Questions

- Compounding
- Discounting
- Determining the rate of return
- Determining the required time to maturity

Introduction to the Bank Account



Symbols

V_0 Present value (i.e. value today)

V_T Future value (i.e. value at time T)

Introduction to the Bank Account



Symbols

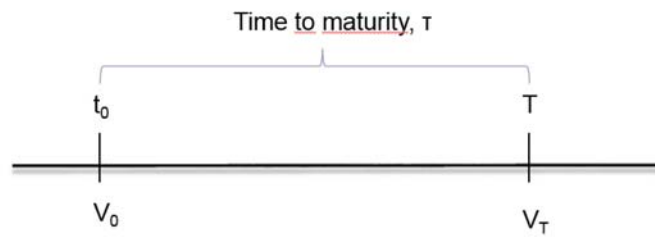
V_0 Present value (i.e. value today)

V_T Future value (i.e. value at time T)

t_0 Today

T Maturity date

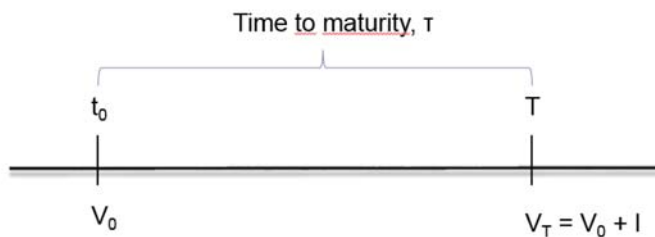
Introduction to the Bank Account



Symbols

V_0	Present <u>value</u> (i.e. <u>value today</u>)
V_T	<u>Future value</u> (i.e. <u>value at time T</u>)
t_0	<u>Today</u>
T	<u>Maturity date</u>
τ	<u>Time to maturity</u> i.e. $(T - t_0)$

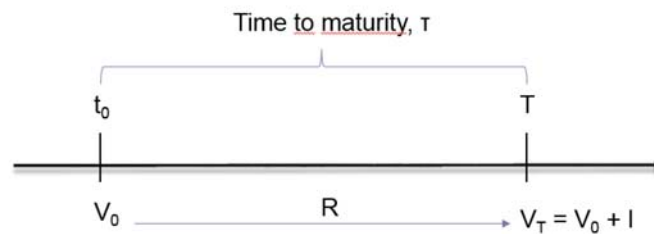
Introduction to the Bank Account



Symbols

V_0	Present <u>value</u> (i.e. <u>value today</u>)
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τ	<u>Time to maturity</u> i.e. $(T - t_0)$
I	<u>Amount of interest earned</u> over the time to maturity

Introduction to the Bank Account



Symbols

V_0	Present value (i.e. value today)
V_T	Future value (i.e. value at time T)
t_0	Today
T	Maturity date
τ	Time to maturity i.e. ($T - t_0$)
I	Amount of interest earned over the time to maturity
R	Rate of return (interest rate expressed as a percentage per annum)

Compounding

Example

Suppose that we invest \$ 100 today at an interest rate of 10% p.a. What will be the future value of our investment within 1 year?

$$100 \times (1 + 10\%) = 110$$

Compounding

Example (2)

Suppose that we invest \$ 100 today at an interest rate of 10% p.a. What will be the future value of our investment within 6 months time?

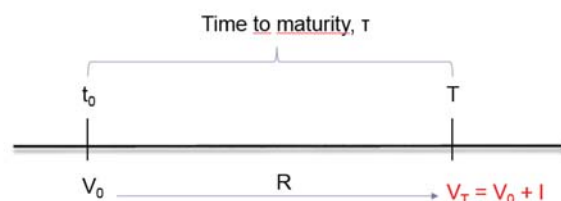
$$100 \times \left(1 + 10\% \times \frac{6}{12}\right) = 105$$

$$100 \times (1 + 10\% \times 0.5) = 105$$

$$V_0 \times (1 + R \times \tau) = V_1$$

Time is measured in years! The interest is expressed per annum.

Recap: Compounding



Symbols

V_0 Present value (i.e. value today)

V_T Future value (i.e. value at time T)

t_0 Today

T Maturity date

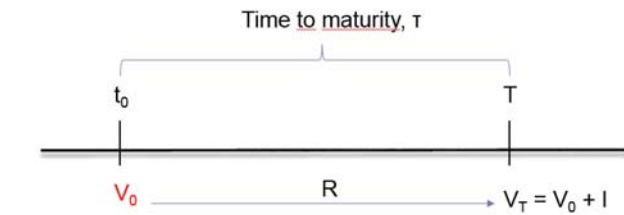
τ Time to maturity i.e. $(T - t_0)$

I Amount of interest earned over the time to maturity

R Rate of return (interest rate expressed as a percentage per annum)

$$V_T = V_0 \cdot (1 + R \cdot \tau)$$

Discounting



Symbols

V_0	Present value (i.e. value today)
V_T	Future value (i.e. value at time T)
t_0	Today
T	Maturity date
τ	Time to maturity i.e. $(T - t_0)$
I	Amount of interest earned over the time to maturity
R	Rate of return (interest rate expressed as a percentage per annum)

$$V_0 = \frac{V_T}{(1 + R \cdot \tau)}$$

Discounting

Example

Suppose that we will receive \$ 110 within one year. What will be the present value of this \$ 110, when the interest rate is 10% p.a.?

Hence

$$\frac{110}{(1 + 10\% \times 1)} = 100$$

Return Calculations

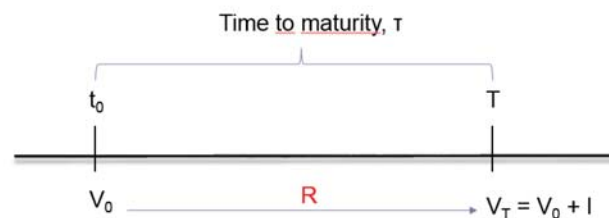
Example

How much return shall I earn if \$ 100 invested today will accrue to \$ 110 over a period of 1 year?

$$V_T = V_0 \times (1 + R \times \tau)$$

$$R = \frac{\frac{V_T}{V_0} - 1}{\tau} = \frac{\frac{110}{100} - 1}{1} = 10\%$$

Return Calculation



Symbols

V_0	Present value (i.e. value today)
V_T	Future value (i.e. value at time T)
t_0	Today
T	Maturity date
τ	Time to maturity i.e. $(T - t_0)$
I	Amount of interest earned over the time to maturity
R	Rate of return (interest rate expressed as a percentage per annum)

$$R = \frac{(V_T/V_0) - 1}{\tau}$$

Time to Maturity Calculation

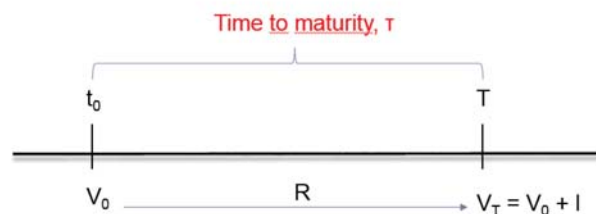
Example

How long will it take to obtain a future value of \$ 110 if I invest \$ 100 today at an interest rate of 10% p.a. ?

$$V_T = V_0 \times (1 + R \times \tau) \text{ implied } R = \frac{\frac{V_T}{V_0} - 1}{\tau}$$

$$\tau = \frac{\frac{V_T}{V_0} - 1}{R} = \frac{\frac{110}{100} - 1}{10\%} = 1$$

Time to Maturity Calculation



Symbols

V_0	Present value (i.e. value today)
V_T	Future value (i.e. value at time T)
t_0	Today
T	Maturity date
τ	Time to maturity i.e. $(T - t_0)$
I	Amount of interest earned over the time to maturity
R	Rate of return (interest rate expressed as a percentage per annum)

$$\tau = \frac{(V_T/V_0) - 1}{R}$$

Two Kinds of Conventions

- Conventions to determine the time to maturity
 - Day Count Conventions
 - Business Day Conventions
- Interest Rate Conventions

Determining the Time to Maturity

The time to maturity is measured **in years** as the ratio of

$$\frac{\text{The number of interest bearing days}}{\text{The number of days in a "year"}}$$

Determining the Time to Maturity

- How much is the future value of \$100 invested for **6 months** at a rate of 10% p.a.?

$$100 \left(1 + 10\% \times \frac{6}{12} \right) = 105$$

- 6 months equals 0.5 year if all months count 30 days and hence a year has 360 days but the actual number of days in a period of six months can be
 - January to June : 181 days 0.49589 year
 - October to March : 182 days 0.49863 year
 - April to September : 183 days 0.50137 year
 - March to August : 184 days 0.50411 year

based on a year of 365 days!

Business Day Conventions

May						
Mo	Tu	We	Th	Fr	Sa	Su
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31
4:○ 11:○ 18:● 25:○						

How long is the time to maturity from May, 4th to May, 8th?

- 4 days

How long is the time to maturity from May, 1st to May 31st?

- 30 days ?

How do we count the days if the contractual payment date falls in a weekend or is a holiday?

Business Day Conventions

Preceding/previous business day: If the contractual date of payment is a day in a weekend or a holiday, the payment will be executed on the last business day before that date.

April	May	June
Mo Tu We Th Fr Sa Su	Mo Tu We Th Fr Sa Su	Mo Tu We Th Fr Sa Su
1 2 3 4 5	1 2 3	1 2 3 4 5 6 7
6 7 8 9 10 11 12	4 5 6 7 8 9 10	8 9 10 11 12 13 14
13 14 15 16 17 18 19	11 12 13 14 15 16 17	15 16 17 18 19 20 21
20 21 22 23 24 25 26	18 19 20 21 22 23 24	22 23 24 25 26 27 28
27 28 29 30	25 26 27 28 29 30 31	29 30
40 12 18 24 30	40 11 18 25	20 9 16 24
July	August	September
Mo Tu We Th Fr Sa Su	Mo Tu We Th Fr Sa Su	Mo Tu We Th Fr Sa Su
1 2 3 4 5	1 2	1 2 3 4 5 6
6 7 8 9 10 11 12	3 4 5 6 7 8 9	7 8 9 10 11 12 13
13 14 15 16 17 18 19	10 11 12 13 14 15 16	14 15 16 17 18 19 20
20 21 22 23 24 25 26	17 18 19 20 21 22 23	21 22 23 24 25 26 27
27 28 29 30 31	24 25 26 27 28 29 30	28 29 30
20 8 15 22 29 31	7 14 22 29	5 13 21 28

April, 4th becomes April, 3rd
 May, 3rd becomes April 30th

Business Day Conventions

Modified preceding business day: In case the preceding business day convention would shift the payment to the previous month, the payment will be shifted to the first business day of the month of the contractual payment date.

April	May	June
Mo Tu We Th Fr Sa Su	Mo Tu We Th Fr Sa Su	Mo Tu We Th Fr Sa Su
1 2 3 4 5	1 2 3	1 2 3 4 5 6 7
6 7 8 9 10 11 12	4 5 6 7 8 9 10	8 9 10 11 12 13 14
13 14 15 16 17 18 19	11 12 13 14 15 16 17	15 16 17 18 19 20 21
20 21 22 23 24 25 26	18 19 20 21 22 23 24	22 23 24 25 26 27 28
27 28 29 30	25 26 27 28 29 30 31	29 30
40 12 18 24 30	40 11 18 25	20 9 16 24
July	August	September
Mo Tu We Th Fr Sa Su	Mo Tu We Th Fr Sa Su	Mo Tu We Th Fr Sa Su
1 2 3 4 5	1 2	1 2 3 4 5 6
6 7 8 9 10 11 12	3 4 5 6 7 8 9	7 8 9 10 11 12 13
13 14 15 16 17 18 19	10 11 12 13 14 15 16	14 15 16 17 18 19 20
20 21 22 23 24 25 26	17 18 19 20 21 22 23	21 22 23 24 25 26 27
27 28 29 30 31	24 25 26 27 28 29 30	28 29 30
20 8 15 22 29 31	7 14 22 29	5 13 21 28

April, 4th becomes April, 3rd
 May, 3rd becomes May, 4th

Business Day Conventions

- 1 Preceding/previous business day
- 2 Modified preceding business day
- 3 Following/next business day
- 4 Modified following business day

Day Count Conventions

How do we count the number of days in the interest bearing period (numerator)?

How do we count the number of days in a year (denominator) ?

Commonly used conventions are

- 1 ACT/ACT
- 2 ACT/365
- 3 ACT/360
- 4 30/360 or 30/360 (American)
- 5 30E/360 or 30/360 (European)
- 6 30E+/360 (European 'Extended')

Day Count Conventions

ACT in the numerator: : Difference between two « serial » dates

- Example 1: Starting date: September, 4th Ending date: September 16th
 - $16 - 4 = 12$ days
- Example 2: Starting date: March, 4th Ending date: September 16th
 - $(31-4) + 30 + 31 + 30 + 31 + 31 + 16 = 196$ days

ACT in the denominator: Complicated algorithm that roughly takes the average days of the years involved (taking into account that a leap year has an extra day).

Day Count Conventions

30/360: How many days are there between D1/M1/Y1 and D2/M2/Y2?

Recipe:

- 1 If the convention is 30/360, 30E/360 or 30E+/360 change D1 in 30 if D1 is 31
- 2 Then
 - 1 30/360: Change D2 in 30 if D2 is 31 AND D1 is either 30 or 31
 - 2 30E/360: Change D2 in 30 if D2 is 31
 - 3 30E+/360: Change D2 in 1 and M2 in M2+1 if D2 is 31
- 3 Compute: $(Y2-Y1) \times 360 + (M2-M1) \times 30 + (D2-D1)$.

Day Count Conventions

30/360 : How many days are there between 31/3/2015 and 31/7/2017?

Recipe:

- ① If the convention is 30/360 change D1 in 30 if D1 is 31
- ② Then 30/360: Change D2 in 30 if D2 is 31 AND D1 is either 30 or 31
- ③ Compute: $(2017-2015) \times 360 + (7-3) \times 30 + (30-30) = 840$ days

Day Count Conventions

30E+/360 : How many days do we count between 31/3/2015 and 31/12/2017?

Recipe:

- ① If the convention is 30E+/360 change D1 in 30 if D1 is 31
- ② Then 30E+/360: Change D2 in 1 and M2 in M2+1 if D2 is 31
- ③ Compute: $(2017-2015) \times 360 + (13-3) \times 30 + (1-30) = 991$ days



Day Count Conventions

- =yearfrac()

Basis	Day count basis
0 or omitted	US (NASD) 30/360
1	Actual/actual
2	Actual/360
3	Actual/365
4	European 30/360

Conclusion

- In order to calculate the time to maturity, we need to know the exact timing of the cash flows (business day conventions) and we need to agree on a day count convention
- We also need to know how long we have to wait in order for the interest earned to accrue to the invested sum?
 - Simple interest rate calculations
 - Compounded interest rate calculations

Future Value

- What would be the future value? Suppose you invest \$100 at 12% p.a. for a period of 5 months without compounding

$$100 \times \left(1 + 12\% \times \frac{5}{12} \right) = 105$$

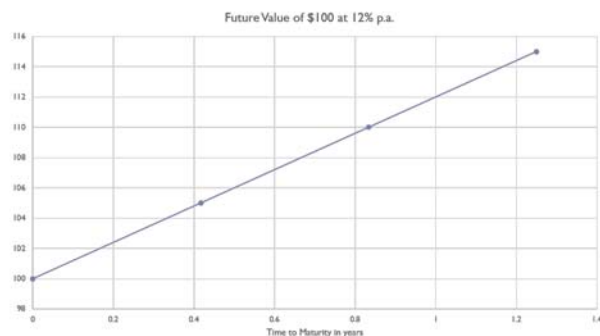
- Suppose you invest \$100 at 12% p.a. for a period of 10 months without compounding

$$100 \times \left(1 + 12\% \times \frac{10}{12} \right) = 110$$

- Suppose you invest \$100 at 12% p.a. for a period of 15 months without compounding

$$100 \times \left(1 + 12\% \times \frac{15}{12} \right) = 115$$

Future Value (Simple Interest)



Reporting Frequency

- Most of the time, the reporting frequency is per annum

$$100 \times (1 + 12\% \text{ p.a.}) = 112$$

$$100 \times \left(1 + 12\% \text{ p.a.} \times \frac{1}{12}\right) = 101$$

$$100 \times (1 + 1\% \text{ p.m.}) = 101$$

- In order to change the reporting frequency, we proportionally adjust the interest rate

$$12\% \text{ p.a.} \Leftrightarrow 3\% \text{ p.q.}$$

Future Value

Which investment do you prefer?

- An investment starting at 10/1/X to 19/1/X of \$ 100 000 renders 5% p.a. (30/360)
- Another investment will give you for the same period an interest rate of 5.05% (ACT/ACT)

Solution

- $100K \times \left(1 + 5\% \times \frac{9}{360}\right) = 100125$

- $100K \times \left(1 + 5.05\% \times \frac{9}{365}\right) = 100124.52$

Future Value

What is interest rate based on ACT/ACT that is equivalent with a 5% p.a. interest rate based on 30/360?

$$100K \times 5\% \times \frac{9}{360} = 100K \times R^* \times \frac{9}{365} \Rightarrow R^* = 5.07\% \text{ (ACT/ACT)}$$

Present Value

What is the present value of \$100 that you will receive in 91 days time if the market interest rate is 5% and the day count convention is ACT/360?

$$PV \times (1 + R \times \tau) = FV \Rightarrow PV = \frac{FV}{(1 + R \times \tau)}$$

$$\frac{100}{(1 + 5\% \times \frac{91}{360})} = 98.7519$$

A First Look at Returns

Suppose that we made a \$100 investment one year ago. Today the value of the investment is \$120.

- $100 \times (1 + R) = 120 \implies R = \frac{120}{100} - 1 = 20\%$
- $R = \frac{V_1}{V_0} - 1 = \frac{V_1 - V_0}{V_0}$
- Simple returns - procentual returns - ordinary returns

Suppose that we made a \$100 investment one month ago. Today the value of the investment is 105.

- $R = \frac{105}{100} - 1 = 5\% \text{ p.m.}$
- $5\% \times 12 = 60\% \text{ p.a.}$

An Interest Rate vs a Discount Rate

Borrow € 800 Repay € 1000

- $R = \frac{200}{800} = 25$
- $d = \frac{200}{1000} = 20\%$

Conversion Formulas

$$V_0 \times (1 + R) = V_1$$

$$V_0 = (1 - d) \times V_1$$

$$\text{so } V_1 = \frac{V_0}{(1-d)}$$

$$\text{Hence } V_0 \times (1 + R) = \frac{V_0}{(1-d)}$$

or

$$(1 + R) = \frac{1}{(1-d)} \implies R = \frac{d}{1-d}$$

$$\implies d = \frac{R}{1+R}$$

An Interest Rate vs a Discount Rate

Suppose we use a discount rate of 5%. The time to maturity is 100 days (ACT/ACT). On \$ 100 000, the discount is \$ 100 000 \times 100/365 \times 5% = \$ 1369.86. The client receives initially \$98 630.14 and has to repay \$100 000 at the maturity date. How much is the implied compounded interest rate (p.a.)?

$$98630 \times \left(1 + R \times \frac{100}{365}\right) = 100000 \implies R = 5.069\% \text{ p.a.}$$

Interim Payments

Suppose an entrepreneur contracts a loan of € 1 000 that needs to be repaid in 12 months at a simple rate of return of 5%. The entrepreneur, however, has the right to interim make repayments. That is why he paid back € 300 after 3 months and another € 250 after 6 months. An interesting and practical question is how much the entrepreneur still needs to pay back at maturity.

Interim Payments (Merchant Rules)

Time in months	Cash Flow	V_T
0	-1000	-1050
3	300	311.25
6	250	256.25
12	?	482.50

Interim Payments (Declining Balance Method)

Time		
0	-1000.00	Starting Balance
3	-12.50	Interest owed
	-1012.50	Total Debt
	300.00	Repayment
	-712.50	New Balance
6	-8.91	Interest owed
	-721.41	Total Debt
	250.00	Repayment
	-471.41	New Balance
12	-11.79	Interest owed
	-483.19	New Balance

Compounded Interest

- Suppose you deposit 1 000 for one year at a rate of 10% p.a.
- How much will it amount to in one year?

$$V_1 = V_0 \times (1 + R) = 1\,000 \times (1.1) = 1\,100$$

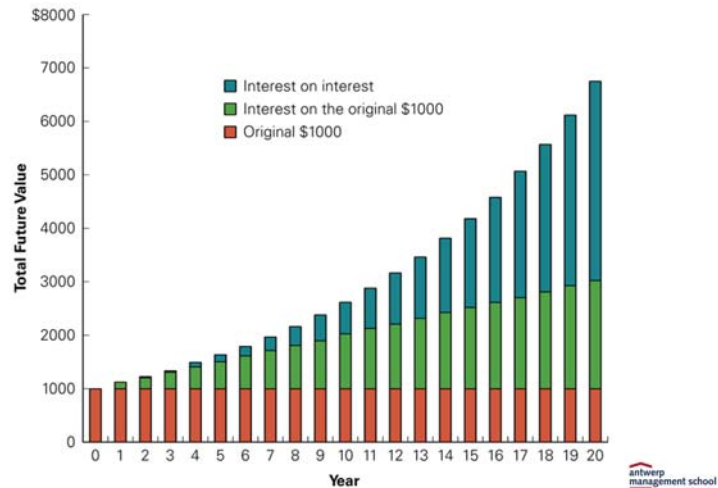
- What happens if you leave it in the account for another year?

$$V_2 = V_1 \times (1 + R) = 1\,100 \times (1.1) = 1\,210$$

- You earned an EXTRA 10 in Year 2 with compound over simple interest.

The Composition of Interest over Time

$$V_T = V_0 \times (1 + R)^T$$



Power of Compounding

The Power of Compounding

Problem

Suppose you invest \$1000 in an account paying 10% interest per year. How much will you have in the account in 7 years? in 20 years? in 75 years?

Source: Berk & Demarzo

Power of Compounding

Solution

You can apply Eq. 4.1 to calculate the future value in each case:

$$\begin{aligned} 7 \text{ years:} & \quad \$1000 \times (1.10)^7 = \$1948.72 \\ 20 \text{ years:} & \quad \$1000 \times (1.10)^{20} = \$6727.50 \\ 75 \text{ years:} & \quad \$1000 \times (1.10)^{75} = \$1,271,895.37 \end{aligned}$$

Note that at 10% interest, your money will nearly double in 7 years. After 20 years, it will increase almost 7-fold. And if you invest for 75 years, you will be a millionaire!

Source: Berk & Demarzo



Compounding

- `=fv(rate; nper; [pmt]; pv; [type])`
- Compounding \$100 at 5% p.a. for a period of 2 years gives
- `=fv(5%; 2; ; 100) = -110.25`

Compounding

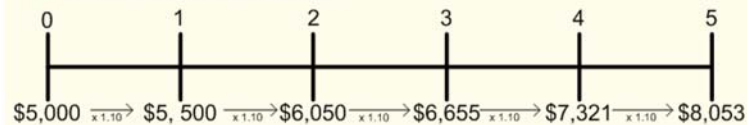
Suppose you have a choice between receiving \$ 5 000 today or \$ 10 000 in five years. You believe you can earn 10% on the \$ 5 000 today, but want to know what the \$ 5 000 will be worth in five years.

Source: Berk & Demarzo

Compounding

■ Solution

- The time line looks like this:



- In five years, the \$5,000 will grow to:

$$\$5,000 \times (1.10)^5 = \$8,053$$
- The future value of \$5,000 at 10% for five years is \$8,053.
- You would be better off forgoing the gift of \$5,000 today and taking the \$10,000 in five years.

Source: Berk & Demarzo

Compounding

Assume a portfolio manager invests € 5 million at a compounded interest rate of 2% p.a. for the first 4 years. The next three years he expects to be able to reinvest the money at 5%. How much will his terminal wealth be after 7 years?

Determining the Time to Maturity and The Rule of 70

- How long does it take to double \$5 000 at a compound rate of 10% per year (approx.)?
- $70 / 10\% = 7$ Years [Actual Time is 7.27 Years]
- $\tau = \frac{\ln\left(\frac{V_T}{V_0}\right)}{\ln(1+R)}$ gives the exact answer.

Determining the Present Value

Example

Allan Hodgson will receive €10 000 three years from now. Allan can earn 8 percent on his investments, so the appropriate discount rate is 8 percent. What is the present value of his future cash flow?

$$PV = \frac{10\,000}{(1 + 8\%)^3} = 7\,938$$

Source: Hillier, Ross, Westerfield, Jaffe & Jordan

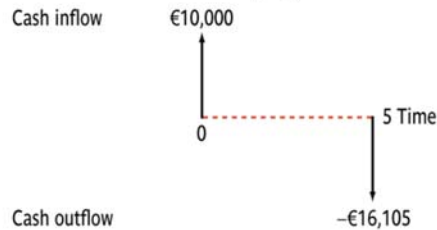
Discounting

- `=pv(rate; nper; [pmt]; fv; [type])`
- Compounding \$100 at 5% p.a. for a period of 2 years gives
- `=pv(5%; 2; ; 110.25) = -100`

Determining the Rate

Example

Carl Voigt, who recently won €10,000 in the lottery, wants to buy a car in five years. Carl estimates that the car will cost €16,105 at that time. His cash flows are displayed below:



What interest rate must he earn to be able to afford the car?



Source: Hillier, Ross, Westerfield, Jaffe & Jordan

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Determining the Rate

Example Cont'd

The ratio of purchase price to initial cash is:

$$\frac{€16.105}{€10.000} = 1.6105$$

Thus, he must earn an interest rate that allows €1 to become €1.6105 in five years.

$$€10,000 \times (1 + r)^5 = €16,105$$

where r is the interest rate needed to purchase the car.

Because $€16,105/€10,000 = 1.6105$, we have:

$$(1 + r)^5 = 1.6105$$

$$r = 10\%$$

$$r = \sqrt[5]{\frac{V_T}{V_0}} - 1$$

Source: Hillier, Ross, Westerfield, Jaffe & Jordan

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Determining the Rate and the Time to Maturity

- `=rate(nper; [pmt]; pv; fv; [type];[guess])`
- `=nper(rate; pmt; pv; fv; [type])`

Higher Frequencies

- Compounding an investment m times a year provides end-of-year wealth of:

$$V_T = V_0 \times \left(1 + \frac{R_m}{m}\right)^{\tau \times m}$$

- where V_0 is the initial investment and R is the stated annual interest rate with m -frequency compounding.

Higher Frequencies

Compounding Frequency	m	Future Value of €100 within 1 year
Yearly	m=1	110
Semi-annually	m=2	110.25
Quarterly	m=4	110.38
Monthly	m=12	110.47
Weekly	m=52	110.5065
Daily	m=365	110.51558

Higher Frequencies

Discounting becomes

$$\frac{V_T}{\left(1 + \frac{R_m}{m}\right)^{\tau \times m}} = V_0$$

Higher Frequencies

Assume an investor is offered a financial instrument that gives him a cash flow of € 1 000 after 1 quarter and another € 2 000 after 2 quarters. How much would the investor be prepared to pay for this instrument if he requires a rate of return of 12% p.a.?

Higher Frequencies

Use Rate/m
Use $N_{\text{per}} \times m$

Continuous Compounding

$$\lim_{m \rightarrow \infty} \left(1 + \frac{R}{m}\right)^{\tau \times m} = e^{(R \times \tau)}$$

$$V_T = V_0 \times e^{r \times \tau}$$

Continuous Compounding

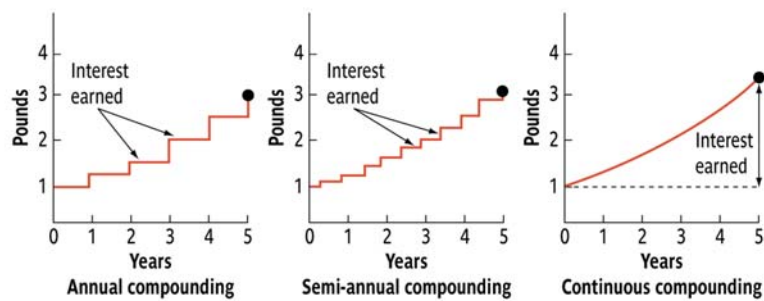
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Quarterly	m=4	110.38
Monthly	m=12	110.47
Weekly	m=52	110.5065
Daily	m=365	110.51558
Continuous Compounding	m=Inf	110.51709



Continuous Compounding

- $=\exp(\text{number})$
- E.g. $100 \times \exp(0.10 \times 1) = 110.51709$

Continuous Compounding



Source:

Continuous Compounding

Discounting becomes

$$V_0 = \frac{V_T}{e^{r \times \tau}}$$

or simply

$$V_0 = V_T \times e^{-r \times \tau}$$

The present value of € 100 due in 3 months at a rate of 4% p.a. c.c. is

$$€ 100 \times \exp(-0.04 \times 0.25) = € 90.005.$$

Conversion Formulas

$$r = m \times \ln\left(1 + \frac{R_m}{m}\right)$$

$$R_m = m \times \left(e^{\frac{r}{m}} - 1\right)$$



Continuous Compounding

- $= \ln(\text{number})$
- E.g. $\ln(1 + 5\%) = 4.88\%$

Discount Factors

The present value of a cash flow that we receive in 3 months time of € 100 is € 97.5. We can express this using

Quarterly Compounding

$$97.5 = \frac{100}{1 + \frac{r_q}{4}} \quad r_q = 10.2564\%$$

Annual Compounding

$$97.5 = \frac{100}{(1 + r_a)^{0.25}} \quad r_a = 10.6577\%$$

Continuous Compounding

$$97.5 = 100 e^{-0.25 \times r_{cc}} \quad r_{cc} = 10.127\%$$

Discount Factor = Present value of a \$1 future payment.

Discount Factors

$$V_0 = V_T \times df(\tau)$$

- $df(m)$ is a non increasing function of τ
- Remark: A discount factor can be interpreted as the price of a zero bond maturing in one years time!

Outline

- 1 Time Value of Money
- 2 Valuing Bonds and Stocks**
- 3 Computing Returns and Return Indices
- 4 Descriptive Statistics and Frequency Distributions
- 5 Theoretical Distributions and Hypothesis Testing
- 6 Portfolio Returns and Risk
- 7 Portfolio Selection
- 8 Univariate Regression
- 9 CAPM and the Single Index Model
- 10 Forwards and Futures
- 11 Introduction to Options

Learning Objectives

- Create time lines for streams of cash flows including bond and stock cash flows
- Recognize annuities
- Derive annuity formulas (constant and growing annuities)
- Given four out of the following five inputs for an annuity, compute the fifth: (a) present value, (b) future value, (c) number of periods, (d) periodic interest rate, (e) periodic payment.
- Given three out of the following four inputs for a single sum, compute the fourth: (a) present value, (b) future value, (c) number of periods, (d) periodic interest rate.
- Given cash flows and present or future value, compute the internal rate of return for a series of cash flows.

Value Additivity

Paul Draper has won a crossword competition and will receive the following set of cash flows over the next two years:

Year	Cash Flow
1	£2,000
2	£5,000

Mr. Draper can currently earn 6 percent in his money market account. What is the present value of the cash flows?

Source: Hillier, Ross, Westerfield, Jaffe & Jordan

Value Additivity

Year 2

$$£5,000 \times \left(\frac{1}{1.06} \right)^2 = £5,000 \times .890 = \underline{£4,450}$$

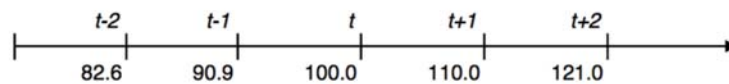
Year 1

$$£2,000 \times \frac{1}{1.06} = £2,000 \times .943 = £1,887$$

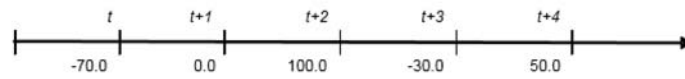
$$\text{Total Value: } £4,450 + £1,887 = £6,337$$

Source: Hillier

The Golden Rule



The Golden Rule



Time	Cash Flow
t	-70
t+1	0
t+2	100
t+3	-30
t+4	50

The Golden Rule

Time	Cash Flow	$V(t)$	$V(t+3)$
t	-70	-70	-93.17
t+1	0	0	0
t+2	100	82.64	110
t+3	-30	-22.54	-30
t+4	50	34.15	45.45
Value of the Project		24.26	32.28

$r = 10\%$



Discounting a Stream of Cash Flows

- =npv(rate,values)

$$NPV = \sum_{i=1}^n \frac{value_i}{(1 + rate)^i}$$

- =xnpv(rate, values, dates)

$$XNPV = \sum_{i=1}^n \frac{value_i}{(1 + rate)^{\frac{date(i) - date(1)}{365}}}$$



Discounting a Stream of Cash Flows

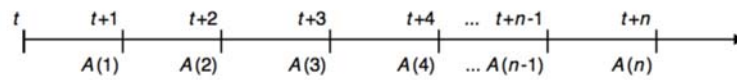
- =npv(0.10;0;100;-30;50) - 70, = 24.26
- =npv(0.10; C2:C5))

Rate	10%
------	-----

1/01/2000	-70
1/01/2001	0
1/01/2002	100
1/01/2003	-30
1/01/2004	50
XNPV	24.23

=XNPV(C2;C4:C8;B4:B8)

The Time Line



Types of Annuities

- ① Constant
- ② Temporary
- ③ Starting immediately
- ④ Ordinary
(Post-numerando)
- ⑤ Simple
- ⑥ Certain

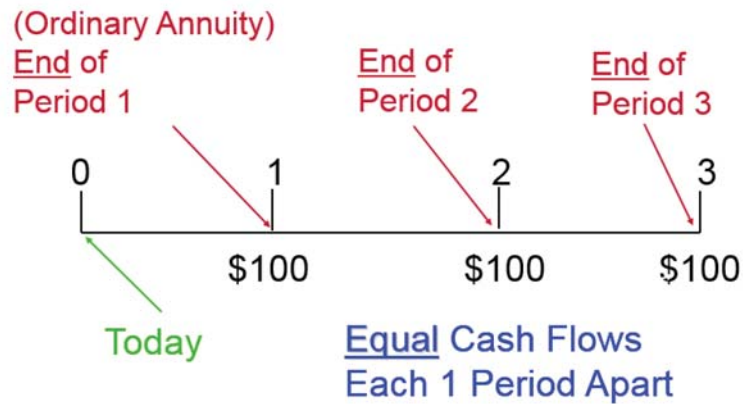
Types of Annuities

- | | |
|--------------------------------|------------------------------------|
| ① Constant | ① Variable |
| ② Temporary | ② Perpetuity |
| ③ Starting immediately | ③ Postponed |
| ④ Ordinary
(Post-numerando) | ④ Annuities due
(Pre-numerando) |
| ⑤ Simple | ⑤ General |
| ⑥ Certain | ⑥ Contingent |

Definitions

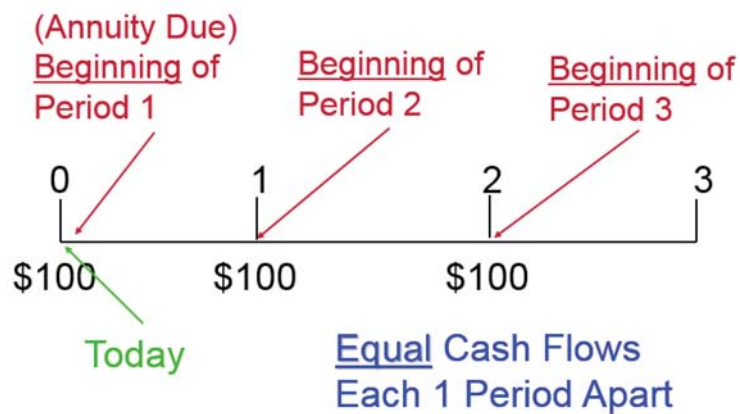
- An **annuity** represents a series of payments (or receipts) occurring over a specified number of equidistant periods
- A **perpetuity** is a stream of cash flows that never ends
- For the **ordinary annuity** the payments or receipts occur at the end of each period
- For the **annuity due** the payments or receipts occur at the beginning of each period
- For **simple annuities** the compounding frequency coincides with the frequency of the cash flows
- **Contingent annuities** have stochastic input parameters

Annuities and their Time Line



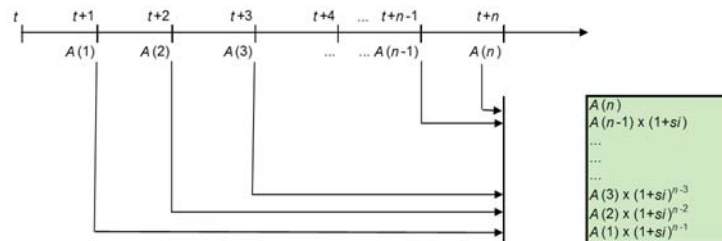
Source: ...

Annuities and their Time Line



Source: ...

The Time Line



By Example

- Suppose on January 1 of year 1 we promise to invest an annual amount of € 10 000 on December 31, every year at an interest rate of 6% p.a (annual compounding)
- After 5 years (on 31 December of year 5), our investor gets € 10 000 $\times 1.064$ + € 10 000 $\times 1.063$ + € 10 000 $\times 1.062$ + € 10 000 $\times 1.06$ + € 10 000 = € 56 371.30
- € 10 000 $\times (1 + 1.06 + 1.062 + 1.063 + 1.064) = € 56 371$
- € 10 000 $\times \left(\frac{1-1.06^4 \times 1.06}{1-1.06} \right) = € 56 371$

Sum of a Geometric Series

$$(1-q) \times (1+q) = 1+q-q-q^2 = 1-q^2;$$

$$(1-q) \times (1+q+q^2) = 1+q+q^2-q-q^2-q^3 = 1-q^3;$$

$$(1-q) \times (1+q+q^2+q^3) = 1+q+q^2+q^3-q-q^2-q^3-q^4 = 1-q^4;$$

...

$$(1-q) \times (1+q+\dots+q^n) = \dots = 1-q^{n+1}.$$

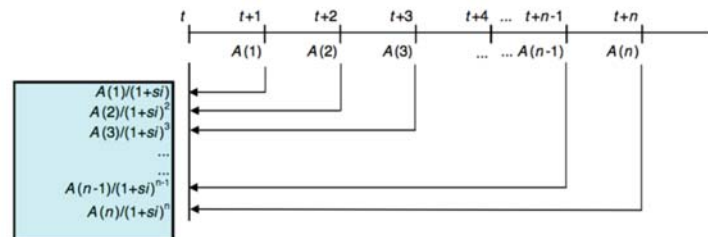
$$1+q+q^2+\dots+q^n = \frac{1-q^{n+1}}{1-q};$$



Exercises

- The argument **Pmt** denotes the fixed payment that will be received/paid for **Nper** periods.
- Suppose an investment manager invests € 5 million in a 8% coupon bond with a remaining time to maturity of 10 years. How much will the terminal wealth be if he can reinvest the coupons at 6.7 % p.a. ?
- **fV(Rate; Nper; Pmt)**
- **fV(0.067; 10; -400 000) = 5 448 885**
- And with semi-annual coupons?
- So what will **fV(6%;4;-10,-20)** compute?

The Time Line



The Maths

$$V_0 = \frac{A}{1+r} + \frac{A}{(1+r)^2} + \dots + \frac{A}{(1+r)^n}$$



$$V_0 = A \times \left[\frac{1 - \left\{ \frac{1}{(1+r)^n} \right\}}{r} \right]$$

or

$$V_0 = A \times \left[\frac{1}{r} - \frac{1}{r(1+r)^n} \right]$$

Period	6%	7%	8%
1	0.943	0.935	0.926
2	1.833	1.808	1.783
3	2.673	2.624	2.577
4	3.465	3.387	3.312
5	4.212	4.100	3.993

Exercise

Suppose an investor is offered a financial instrument at a price of € 5 300. The instrument will give a cash flow of € 500 at the end of each of the next 20 years. Assume the investor requires a rate of return of 5.5% p.a. Please advise?

Parameter Search

$$A = \frac{V_0}{\left[\frac{1 - \frac{1}{(1+r)^n}}{r} \right]}$$

$$pv \times (1 + rate)^{nper} + pmt \times \left(\frac{(1 + rate)^{nper} - 1}{rate} \right) + fv = 0$$

Note: (Pmt * Nper) + Pv + Fv = 0 if rate = 0

Parameter Search

- Rippoff nv advertizes that you can get a loan at 8% (simple interest rate). The time to maturity is 18 months. You will have to finance € 2 400.
- The future value is $€ 2\,400 \times (1 + 8\% \times 18/12) = € 2\,688$.
Consequently Rippoff says, the monthly downpayment equals $€ 2\,688/18 = € 149.34$.
- Quid?



Parameter Search Examples

- $\text{pmt}(6\%; 4; -20; 69) = -10$
- $\text{nper}(5.5\%; -500; 5975.19) = 20$
- $\text{nper}(6\%; 10; 20; -71.62; 1) = 3.9999676 \approx 4$
- $\text{rate}(18; -149.34; 2400) = 1.22169\%$ (of 14.66% p.a.)

Parameter Search: IRR

In some situations, you know the present value and cash flows of an investment opportunity but you do not know the internal rate of return (IRR), the interest rate that sets the net present value of the cash flows equal to zero.

Computing the Internal Rate of Return with the Annuity Spreadsheet in Excel

Problem

Jessica has just graduated with her MBA. Rather than take the job she was offered at a prestigious investment bank—Baker, Bellingham, and Botts—she has decided to go into business for herself. However, Baker, Bellingham, and Botts was so impressed with Jessica that it has decided to fund her business. In return for an initial investment of \$1 million, Jessica has agreed to pay the bank \$125,000 at the end of each year for the next 30 years. What is the internal rate of return on Baker, Bellingham, and Botts's investment in Jessica's company, assuming she fulfills her commitment?

Source: Berk and Demarzo

Parameter Search: IRR

Solution

Here is the timeline (from Baker, Bellingham, and Botts's perspective):



The timeline shows that the future cash flows are a 30-year annuity. Setting the NPV equal to zero requires

$$1,000,000 = 125,000 \times \frac{1}{r} \left(1 - \frac{1}{(1+r)^{30}} \right)$$

Using the annuity spreadsheet to solve for r ,

	NPER	RATE	PV	PMT	FV	Excel Formula
Given	30		-1,000,000	125,000	0	
Solve for Rate		12.09%				=RATE(30,125000,-1000000,0)

The IRR on this investment is 12.09%.

Source: Berk and Demarzo

Parameter Search: IRR

Computing the Internal Rate of Return Directly

Problem

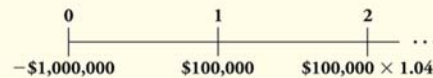
Baker, Bellingham, and Botts offers Jessica a second option for repayment of the loan. She can pay \$100,000 the first year, increase the amount by 4% each year, and continue to make these payments forever, rather than for 30 years. What is the IRR in this case?

Source: Berk and Demarzo

Parameter Search: IRR

Solution

The timeline is



The timeline shows that the future cash flows are a growing perpetuity with a growth rate of 4%. Setting the NPV equal to zero requires

$$1,000,000 = \frac{100,000}{r - 0.04}$$

We can solve this equation for r

$$r = 0.04 + \frac{100,000}{1,000,000} = 0.14$$

The IRR on this investment is 14%.

Source: Berk and Demarzo

Perpetuities

Years	Cash Flow	Present Value	Value of the Annuity until year n
1	100.00 €	90.91 €	90.90 €
5	100.00 €	62.09 €	379.08 €
10	100.00 €	38.55 €	614.46 €
20	100.00 €	14.86 €	851.36 €
50	100.00 €	0.85 €	991.48 €
∞	100.00 €	- €	1,000.00 €

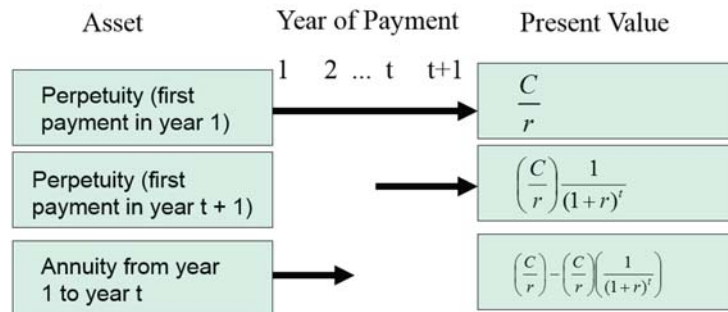
Perpetuities

$$\lim_{n \rightarrow \infty} A \times \left[\frac{1 - \left\{ \frac{1}{(1+r)^n} \right\}}{r} \right]$$



$$V_0 = \frac{A}{r}$$

Annuities are a Portfolio of Two Perpetuities



Annuities Due

- Mark Young receives £50 000 a year for 20 years from a competition. Assume that the first payment occurs immediately and that the discount rate is 8 percent. What is the value of the prize?

$$\begin{aligned}
 &\underbrace{\text{£50,000}}_{\text{Payment at date 0}} + \underbrace{\text{£50,000} \times A_{0.08}^{19}}_{\text{19-year annuity}} \\
 &= \text{£50,000} + (\text{£50,000} \times 9.6036) \\
 &= \text{£530,180}
 \end{aligned}$$

Source: Hillier, Ross, Westerfield, Jaffe & Jordan

Growing Annuities

$$V_0 = A \times \left[\frac{1 - \left(\frac{1+g}{1+r} \right)^n}{r - g} \right]$$

Growing Annuities

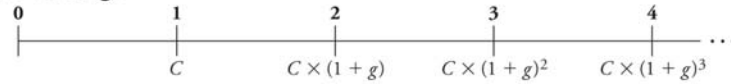
- Stuart Gabriel, a second-year MBA student, has just been offered a job at £80 000 a year. He anticipates his salary increasing by 9 percent a year until his retirement in 40 years. Given an interest rate of 20 percent, what is the present value of his lifetime salary?

- $£80\,000 \times \left[\frac{1 - \left(\frac{1+0.09}{1+0.20} \right)^{40}}{0.20 - 0.09} \right] = £71\,1730.71$

Source: Hillier, Ross, Westerfield, Jaffe & Jordan

Growing Perpetuities

Assume you expect the amount of your perpetual payment to increase at a constant rate, g .



$$\lim_{n \rightarrow \infty} A \times \left[\frac{1 - \left(\frac{1+g}{1+r} \right)^n}{r - g} \right] = \frac{A}{r - g}$$

Annuities are a Portfolio of Two Perpetuities

	Year: 1	2	3	4	5	6 ...	Present value
1. Growing perpetuity A	\$1	\$1 x (1 + g)	\$1 x (1 + g) ²	\$1 x (1 + g) ³	\$1 x (1 + g) ⁴	\$1 x (1 + g) ⁵ ...	$\frac{1}{r - g}$
2. Growing perpetuity B				\$1 x (1 + g) ³	\$1 x (1 + g) ⁴	\$1 x (1 + g) ⁵ ...	$\frac{1}{(r - g)(1 + r)^3}$
3. Growing 3-year annuity (1 - 2)	\$1	\$1 x (1 + g)	\$1 x (1 + g) ²				$\frac{1}{r - g} - \frac{1}{(r - g)(1 + r)^3}$

Source: Hillier, Ross, Westerfield, Jaffe & Jordan

How Many Mistakes Can You Make?

- Pindado SA is just about to pay a dividend of €3.00 per share. Investors anticipate that the annual dividend will rise by 6 percent a year forever. The applicable discount rate is 11 percent. What is the share price today?

- $3 + \frac{3.18}{0.11 - 0.06} = 66.60$

Source: Hillier, Ross, Westerfield, Jaffe & Jordan

Delayed Annuities

Danielle Caravello will receive a four-year annuity of €500 per year, beginning at date 6. If the interest rate is 10 percent, what is the present value of her annuity? How do you do it?



1. Discount annuity back to year 5
2. Discount year 5 value of annuity back to year 0

Source: Hillier, Ross, Westerfield, Jaffe & Jordan

Delayed Annuities

Step 1: Discount annuity to year 5

$$\text{€}500 \left[\frac{1 - \frac{1}{(1.04)^4}}{.10} \right] = \text{€}500 \times 3.1699$$

$$= \text{€}1,584.95$$

Step 2: Discount year 5 value back to year 0

$$\frac{\text{€}1,584.95}{(1.10)^5} = \text{€}984.13$$

Source: Hllier, Ross, Westerfield, Jaffe & Jordan

Delayed Annuities

Suppose we will pay 5 annual payments of €100. The first payment will be due in two years time. How much is the present value of this delayed annuity at an interest rate of 4% p.a. ?

- $\text{pv}(4\%; 5; -100)/1.04$
- $\text{pv}(4\%; 1; ; -\text{pv}(4\%; 5; -100))$
- $\text{pv}(4\%, 2;; -\text{pv}(4\%; 5; -100; ; 1))$

Amortization Table

- Suppose we borrow € 1 000 at 6% p.a. a.c. for a period of 4 years. We repay a fixed amount per year at the end of the year (covering both the repayment of the principal and the interest expenses).

Amortization Table

Pmt	-288.59 €	=PMT(6%;4;1000)
Ipmt	-46.28 €	=IPMT(6%;2;4;1000)
Ppmt	-242.31 € -256.85 €	=PPMT(6%;2;4;1000)
Cumipmt	-46.28 € -106.28 € -154.37 €	=CUMIPMT(6%;4;1000;2;2;0) =CUMIPMT(6%;4;1000;1;2;0) =CUMIPMT(6%;4;1000;1;4;0)
Cumprinc	-727.74 €	=CUMPRINC(6%;4;1000;1;3;0)

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Learning Objectives

- Know the calculation rules for ordinary and logarithmic returns
- Know the dollar weighted and time weighted average returns
- Know the ways value weighted, equally weighted and price weighted indices are constructed

Ordinary vs Logarithmic Returns

- **Ordinary returns (simple returns)** are based on the simple one period interest calculation
- **Logarithmic returns** are based on the concept of continuous compounding

Ordinary Returns

Let P_t be the price of a non-dividend paying asset at time t

- We know from the rules of the simple interest that
 $P_{t+1} = P_t \times (1 + r_{t+1})$
- Gross returns are price relatives

$$R_{t+1} = (1 + r_{t+1}) = \frac{P_{t+1}}{P_t}$$

- Net returns are percentage changes

$$r_{t+1} = \frac{P_{t+1}}{P_t} - 1 = \frac{P_{t+1} - P_t}{P_t}$$

Ordinary Returns

Let P_t be the price of a dividend paying asset at time t

- Terminal wealth at the end of a period consists out of the final value of the asset + the income received in that period
- Gross returns

$$R_{t+1} = (1 + r_{t+1}) = \frac{P_{t+1} + D_{t+1}}{P_t}$$

- Net returns

$$r_{t+1} = \frac{P_{t+1} + D_{t+1} - P_t}{P_t}$$

- Return Calculations also have to take into account stock splits and other corporate events.

Logarithmic Returns (aka Continuous Compounded Returns)

Let P_t be the price of a non-dividend paying asset at time t

We know from the rules of the continuous compounded interest that

$$P_{t+1} = P_t e^{r_{t+1}^c}$$

Hence logarithmic returns are defined as

$$r_{t+1}^c = \ln\left(\frac{P_{t+1}}{P_t}\right) = \ln(R_{t+1}) = \ln(1 + r_{t+1})$$

Which returns to choose

- **Ordinary returns** : Whenever you have to form portfolios.
- **Logarithmic returns**: For pure time series issues.

Returns in the cross-section

Let r_i be the ordinary return of asset i , n the number of assets in the portfolio and w_i the weight of asset i in the portfolio. The **return of the portfolio** equals the weighted average return of the various assets:

$$r_{p,t+1} = \sum_{i=1}^n w_{i,t} r_{i,t+1}$$

Measuring multi-period returns

$$P_{t+n} = P_0 \times (1 + r_{t+1}) \times (1 + r_{t+2}) \times \dots \times (1 + r_{t+n})$$

Alternatively, we can write

$$P_{t+n} = P_t \times (1 + r_{avg})^n$$

Consequently,

$$(1 + r_{avg})^n = (1 + r_{t+1}) \times (1 + r_{t+2}) \times \dots \times (1 + r_{t+n})$$

or

$$r_{avg} = \sqrt[n]{(1 + r_{t+1}) \times (1 + r_{t+2}) \times \dots \times (1 + r_{t+n})} - 1$$

This is the geometric average of the ordinary returns

Measuring multi-period returns

$$r_{avg} = \sqrt[n]{(1 + r_{t+1}) \times (1 + r_{t+2}) \times \dots \times (1 + r_{t+n})} - 1$$

In terms of the price relatives we obtain

$$r_{avg} = \sqrt[n]{\frac{P_{t+1}}{P_t} \times \frac{P_{t+2}}{P_{t+1}} \times \dots \times \frac{P_{t+n}}{P_{t+n-1}}} - 1 = \sqrt[n]{\frac{P_{t+n}}{P_t}} - 1$$

So in order to estimate a multi-period growth rate

- you only need two prices: the one at the beginning of the period and the one at the end of the period
- a higher frequency of data will not be informative

What about the Arithmetic Average of the Ordinary Returns?

- Estimates the '(unconditional) expected return' in a naive way
- Does not measure the performance of a specific portfolio

Performance Measurement

The **geometric average return** or the **time weighted average return** measures

- the constant return that yields the same total return over a longer period
- does measure performance but ... assigns the same weight to each period.
- is not useful if there are cash inflows and/or cash outflows in the portfolio

The **internal rate of return** or the **dollar weighted average return**

- measures the performance of a portfolio
- taking into account the total amounts invested in each period

The Dollar Weighted Average Return

- $P_t = 50$
- $P_{t+1} = 53$
- $d_{t+1} = 2$

$$\Rightarrow 50 = \frac{53 + 2}{1 + r}$$

$$\Rightarrow r = 10\%$$

The Dollar Weighted Average Return

Example

- Suppose that you purchase 1 share at time t at €50
- Suppose that you purchase at time $t + 1$ another share at €53
- At time $t + 1$ you also receive a dividend of €2 from the first share
- At time $t + 2$ the share price has increased to €54 and you receive a dividend of €2 per share

To compute the dollar weighted average return we equate

$$50 + \frac{53}{1 + r} = \frac{2}{(1 + r)} + \frac{112}{(1 + r)^2}$$

$$\Rightarrow r_{DW} = 7.117\%$$

The Time Weighted Average Return

The time weighted average return equals

$$r_{t+1} = \frac{53 + 2 - 50}{50} = 10\% \text{ and } r_{t+2} = \frac{54 + 2 - 53}{53} = 5.66\%$$

$$\Rightarrow r_{TW} = \sqrt{(1 + 10\%) \times (1 + 5.66\%) - 1} = 7.83\%$$

Concept Check

Shares of XYZ pay €2 dividend at the end of every year on December 31. An investor buys two shares of the stock on January, 1 at a price of €20 each, sells one of those shares for €22 a year later on the next January, 1 and sells the second share an additional year later for €19. Find the time and dollar weighted average returns on this two year investment.

Concept Check - Solution

The Dollar weighted return

$$40 = \frac{22 + 4}{(1 + r)} + \frac{19 + 2}{(1 + r)^2} \Rightarrow r_{DW} = 11.91\%$$

The time weighted return

$$r_{t+1} = \frac{(24 - 20)}{20} = 20\% \text{ and } r_{t+1} = \frac{(21 - 22)}{22} = -4.55\%$$

Hence

$$\Rightarrow r_{avg} = \sqrt{(1 + 20\%) \times (1 - 4.55\%)} - 1 = 7.03\%$$

Data

Assume the market consists out of 3 stocks

Stock	P_0	P_1	Number of Stocks
A	150	150	50
B	40	80	100
C	10	30	500

Construction of a Value Weighted Index

Let's denote the market value at time t as MV_t , then

- $MV_0 = 150 \times 50 + 40 \times 100 + 10 \times 500 = 16\,500$
- $MV_1 = 150 \times 50 + 80 \times 100 + 30 \times 500 = 30\,500$

If we set the index to basis 100 at time 0: $I_0 = 100$
 then $I_1 = \frac{MV_1}{MV_0} \times I_0 = 184.85$

Construction of a Value Weighted Index

Notice that the return on the market $\frac{I_1 - I_0}{I_0} = 84.85\%$ and

Stock	w_0	r_1
A	45%	0%
B	24%	100%
C	30%	200%

Consequently $r_p = 45\% \times 0\% + 24\% \times 100\% + 30\% \times 200\% = 84.85\%$

Construction of an Equally Weighted Index

Stock	w_0	r_1
A	33.33%	0%
B	33.33%	100%
C	33.33%	200%

Consequently

$$r_p = 33.33\% \times 0\% + 33.33\% \times 100\% + 33.33\% \times 200\% = 100\%$$

Construction of a Equally Weighted Index

How does the investment strategy look like?

- Let's invest 1 000 in each stock, so that we can buy 6.667 shares of A, 25 shares of B and 100 shares of C then
- $MV_0 = 3\,000$
- $MV_1 = 150 \times 6.667 + 80 \times 25 + 30 \times 100 = 6\,000$
- If we set the index to basis 100 at time 0: $I_0 = 100$
- then $I_1 = \frac{MV_1}{MV_0} \times I_0 = 200$ yielding $\frac{I_1 - I_0}{I_0} = 100\%$

Constructing a Price Weighted Index

Recall

Stock	P_0	w_0	r_1
A	150	75%	0%
B	40	20%	100%
C	10	5%	200%

Consequently $r_p = 75\% \times 0\% + 20\% \times 100\% + 5\% \times 200\% = 30\%$

Alternatively

- $I_0 = 150 + 40 + 10 = 200$
- $I_1 = 150 + 80 + 30 = 260$
- $\frac{I_1 - I_0}{I_0} = 30\%$

Constructing a Price Weighted Index

The **Dow Jones Industrial Average** is a price weighted index

A scaling factor is used to take change in index composition into account

Stock	P_0
A	$19 \frac{5}{8}$
B	27
C	$52 \frac{1}{2}$

Assume the scaling factor (aka the divisor) equals $d_0 = 2.6$

The index at time 0 equals $I_0 = \frac{19.625 + 27 + 52.5}{2.6} = 38.125$

Constructing a Price Weighted Index

Assume that at the end of day 0, share A is taken over and will be replaced by share D which is priced at 39.

The scaling factor will be adjusted accordingly: $\frac{39+27+52.5}{d_2} = 38.125$ yielding $d_2 = 3.108197$.

Stock	P_0	P_1
A	19 5/8	
B	27	27 1/2
C	52 1/2	51 1/4
D	39	39 1/2

The index at time 1 equals $I_1 = \frac{27.5+51.25+39.5}{3.108197} = 37.924$

Constructing a Price Weighted Index

Assume that during the second day, company C splits 3 for 2

The equivalent price at time 1 for stock C equals $2 \times 51.25 = 3 \times \hat{P}_1$ yielding $\hat{P}_1 = 34.16667$

The scaling factor will be adjusted accordingly: $\frac{39.5+25.5+34.16667}{d_3} = 37.924$ yielding $d_3 = 2.65773$.

Stock	P_0	P_1	P_2
A	19 5/8		
B	27	27 1/2	28
C	52 1/2	51 1/4	34 1/2
D	39	39 1/2	40

The index at time 2 equals $I_1 = \frac{40+28.125+34.5}{2.65773} = 38.614$

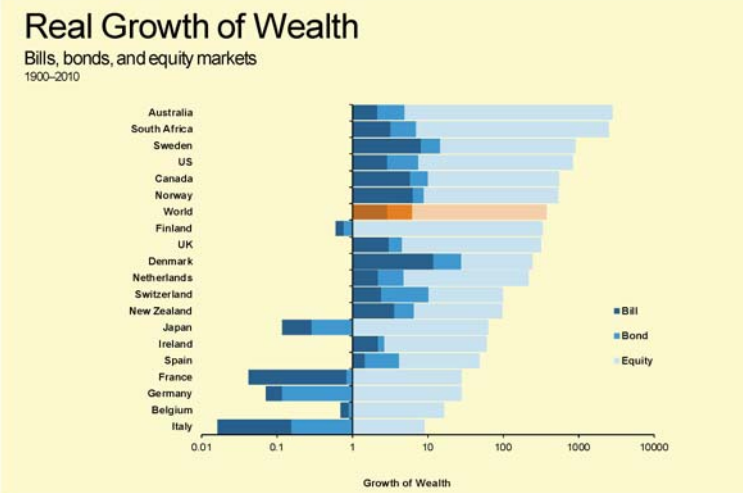
Outline

- 1 Time Value of Money
- 2 Valuing Bonds and Stocks
- 3 Computing Returns and Return Indices
- 4 Descriptive Statistics and Frequency Distributions**
- 5 Theoretical Distributions and Hypothesis Testing
- 6 Portfolio Returns and Risk
- 7 Portfolio Selection
- 8 Univariate Regression
- 9 CAPM and the Single Index Model
- 10 Forwards and Futures
- 11 Introduction to Options

Motivation

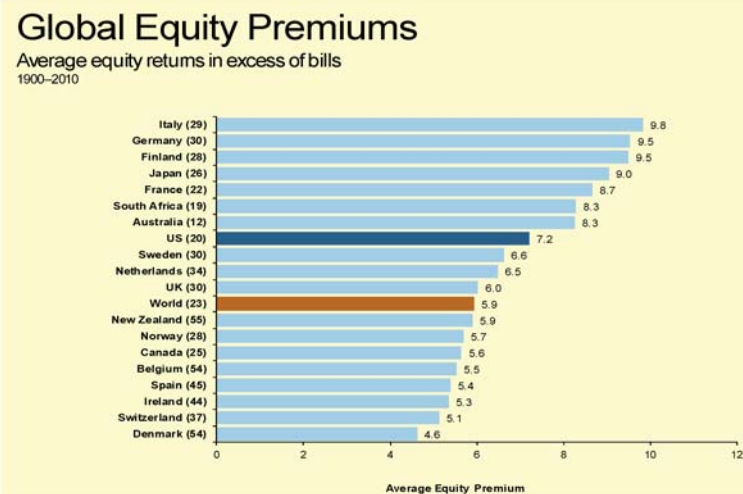
- Do small stocks outperform large stocks?
- By how much do equity investments outperform bond investments?
- Is a hedge fund strategy riskier than a buy-and-hold strategy?
- How do we measure risk?
- How is portfolio risk affected by adding a given stock?
- How are options priced?
- What is the probability of default of a given company?
- What is the probability of default of a portfolio mortgage loans?
- Does a manager possess market timing skills? Sell in May and go away?
- Do commodity investments protect against inflation?
- ...

Real Growth of Wealth



Source: Lee, M., 2012, A Century of Global Returns, Dimensional Fund Advisors. Based on data by Dimson, Marsh & Staunton (DMS).

Global Equity Premiums

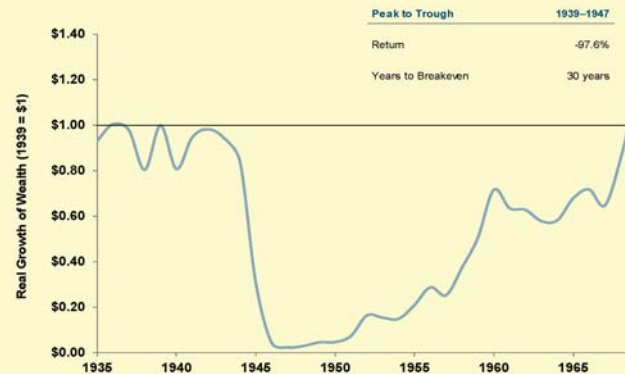


Source: Lee, M., 2012, A Century of Global Returns, Dimensional Fund Advisors. Based on data by Dimson, Marsh & Staunton (DMS).

Worst-Performing Periods for Equities

Worst-Performing Period for Equities

Real growth of wealth: Japan
1935–1969



Source: Lee, M., 2012, A Century of Global Returns, Dimensional Fund Advisors. Based on data by Dimson, Marsh & Staunton (DMS).

Worst-Performing Periods for Equities

Worst-Performing Periods for Equities

1900–2010

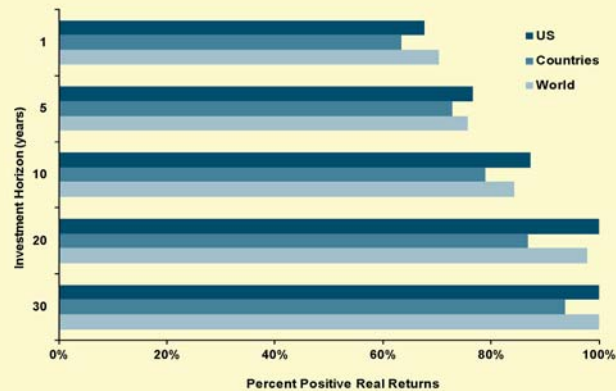
	Peak to Trough	Real Total Return (%)	Total Inflation (%)	Annualized Real Return (%)	Annualized Inflation (%)	Years to Breakeven
Japan	1936–1947	-97.58	7,417	-28.70	48.10	33
Germany	1913–1948	-95.37	2.15E+14	-8.41	125.08	45
France	1942–1950	-87.53	879	-22.91	33.00	43
Italy	1961–1977	-85.74	229	-11.46	7.72	37
Finland	1916–1921	-85.21	667	-31.77	50.31	19
Spain	1973–1982	-84.37	313	-18.63	17.08	23
Belgium	1941–1948	-79.30	41	-20.15	5.00	31
Ireland	2006–2008	-75.06	6	-50.08	2.91	—
Norway	1916–1921	-73.68	75	-23.43	11.81	19
New Zealand	1986–1990	-72.96	34	-27.89	7.66	17
Switzerland	1914–1920	-72.52	121	-19.37	14.10	13
UK	1972–1974	-70.61	32	-45.78	14.78	11
Sweden	1916–1920	-69.03	92	-25.40	17.64	20
Australia	1969–1974	-65.94	47	-19.38	7.96	16
US	1928–1931	-59.52	-15	-26.03	-5.20	8
Netherlands	1969–2008	-56.40	21	-8.58	2.13	11
Canada	1928–1932	-55.32	-20	-18.24	-5.48	7
South Africa	1919–1920	-52.23	47	-52.23	47.46	4
Denmark	2007–2008	-49.19	2	-49.19	2.43	—
World	1928–1931	-53.27	-15	-22.40	-5.20	7

Source: Lee, M., 2012, A Century of Global Returns, Dimensional Fund Advisors. Based on data by Dimson, Marsh & Staunton (DMS).

Are Stocks Less Risky in the Long Run

Are Stocks Less Risky in the Long Run?

Percent of rolling periods with positive real returns



Source: Lee, M., 2012, A Century of Global Returns, Dimensional Fund Advisors. Based on data by Dimson, Marsh & Staunton (DMS).

Random Variables (RV)

- RV attaches a **numerical value** to an experiment's outcome.
- They describe the "state of nature" (cf. "state variable").
- We need:
 - Possible outcomes → **outcome space/range**.
 - Probabilities for each outcome → **probability distribution function** (pdf).

Example: Tossing a Die

Experiment = 'cast a die'.

- RV = numerical value = value rolled with die.
- If we know the value, we know which side of the die shows → the value describes the state of nature.
- Outcome space: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- Probability for each outcome = $1/6$.

Probability Distribution Function

- A pdf $p(x) = \mathbb{P}[X = x]$ satisfies the following properties:
 - 1 Probabilities are non-negative.
 - 2 Probabilities sum to one (certainty).
 - 3 The probability of an event, i.e. a subset of the outcome space, is the sum of the probabilities of all outcomes in the subset.
- For the die:
 - 1 $p(x) = 1/6 > 0$ for $x = 1, 2, \dots, 6$.
 - 2 $p(1) + p(2) + \dots + p(6) = 6 \times \frac{1}{6} = 1$.
 - 3 Let event $E =$ cast an odd number: $E = \{1, 3, 5\}$
→ probability of $E = \mathbb{P}[E] = p(1) + p(3) + p(5) = 0.5$.

Cumulative Distribution Function

- An **alternative representation** is the **cdf**: it gives the probability to obtain an outcome **at most** equal to a given value: $F(x)$ = probability RV is at most x : $F(x) = \mathbb{P}[X \leq x]$.
- For the die:
 - $F(1) = 1/6 = p(1)$
 - $F(2) = 2/6 = p(1) + p(2)$
 - $F(3) = 3/6 = p(1) + p(2) + p(3)$
 - $F(4) = 4/6 = p(1) + p(2) + p(3) + p(4)$
 - $F(5) = 5/6 = p(1) + p(2) + p(3) + p(4) + p(5)$
 - $F(6) = 6/6 = p(1) + p(2) + p(3) + p(4) + p(5) + p(6)$
- Can easily be extended to values outside the outcome space, e.g.:
 - $F(0) = 0 = \mathbb{P}[\emptyset]$ (the empty set, the impossible event)
 - $F(3.5) = 3/6 = p(1) + p(2) + p(3)$
 - $F(100) = 1 = p(1) + p(2) + p(3) + p(4) + p(5) + p(6)$

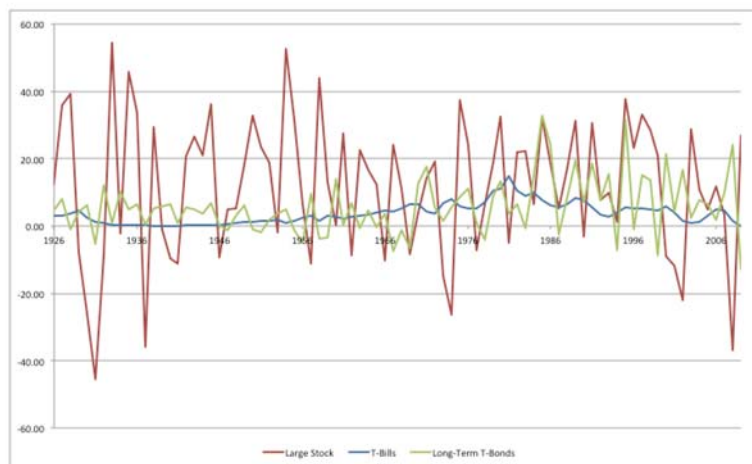
pdf versus cdf

- From **pdf** to **cdf**:
 - Simply **sum** probabilities.
 - cdf is probability of the event $E = \{X \leq x\}$.
 - E.g. $F(3) = \mathbb{P}[E] = p(1) + p(2) + p(3) = 0.5$
- From **cdf** to **pdf**:
 - Simply **subtract** cumulative probabilities.
 - E.g. $p(3) = F(3) - F(2) = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$.
 - $p(4.5) = F(4.5) - F(4) = \frac{4}{6} - \frac{4}{6} = 0$.
 - Hey! We've extended the pdf to outcomes outside the outcome space...

Returns as Random Variables

- Returns interpreted as the outcome of an abstract experiment. Their values are outcomes of a RV.
- They describe the “state of nature” (cf. “state variable”). Their outcomes tell us something about the economy.
- We need:
 - Possible outcomes → **outcome space/range**.
 - Probabilities for each outcome → **probability distribution function** (pdf).
- How?
 - Historical.
 - Theoretical.
- How to convey information? (description)

US Assets



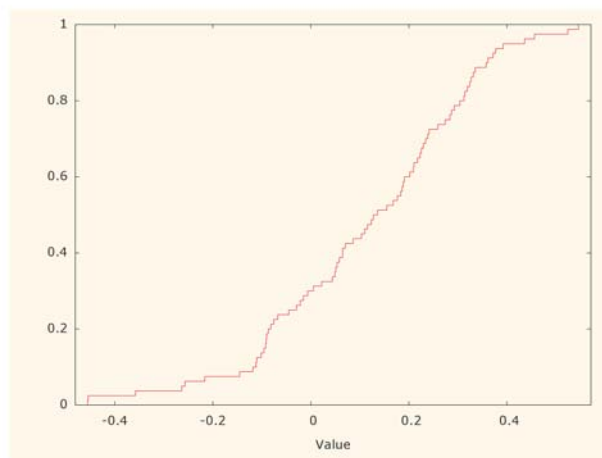
The Discrete Uniform Distribution

- One way to represent this RV, is using the **uniform distribution**. For a uniform distribution, all outcomes are equally likely (hence, **uniform**):

$$\begin{cases} p(x) = \frac{1}{N}, & x \in \{x_1, x_2, \dots, x_N\} \\ = 0 & \text{otherwise} \end{cases}$$

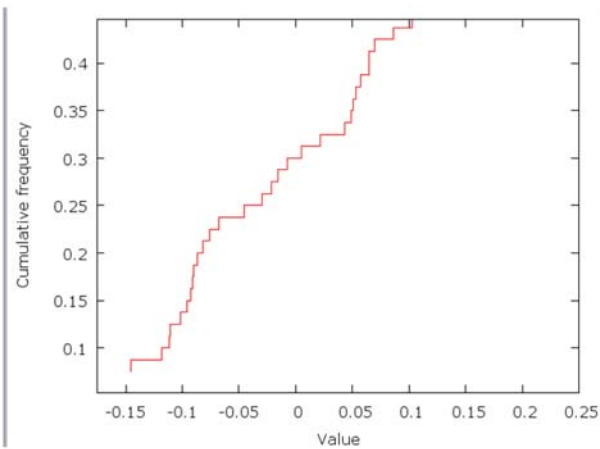
- It is called the **empirical distribution function**.
- It results in the following cdf.

The Cumulative Empirical Distribution Function



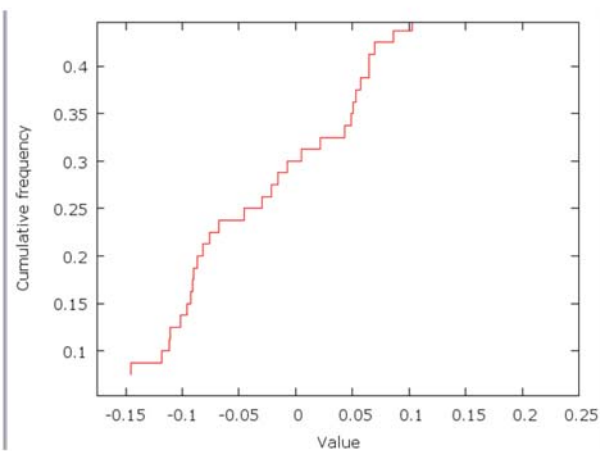
Note the jumps at the different observations!

Let's Zoom In



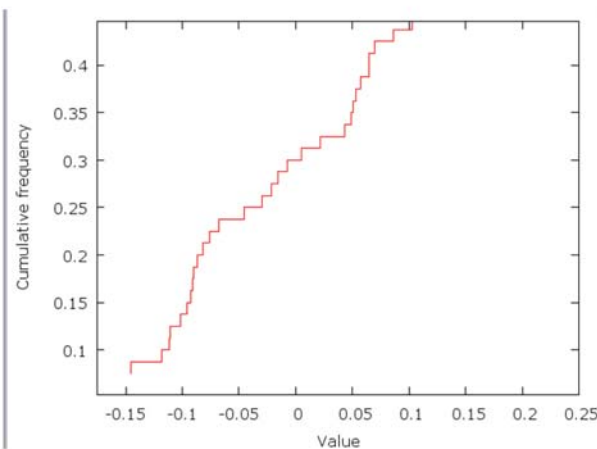
- What is the probability of a negative return?

Let's Zoom In



- What is the probability to have a return between -5% and +5%?

Let's Zoom In



- What is the probability to have a return between 0% and +10%?

Problems Uniform Distribution

- But do we really think that the outcomes that have been observed historically are **all** possible outcomes?
 - Do all other outcomes really have probability zero?
- The empirical distribution function is not necessarily stable. When we add some observations, all probabilities change.
- A cdf is not handy to convey information.

Sorting and Counting: Frequency Distributions

- Another way to describe the historical sample is to
 - Sort the data.
 - Count how many observations fall in pre-specified intervals.
 - Assume probability is spread evenly within each interval → accounts to some extent for values not (yet) observed.
- Issues:
 - How many intervals?
 - Absolute versus relative frequencies.
 - Cumulative frequencies?

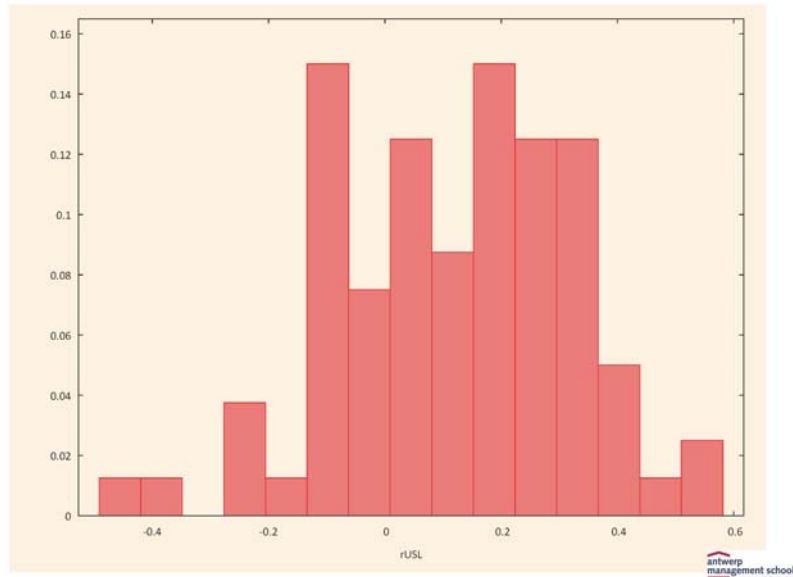
US Large Stocks

Frequency distribution for rUSL, obs 1-81
 number of bins = 15, mean = 0.12145, sd = 0.202593

interval	midpt	frequency	rel.	cum.
< -0.41984	-0.45560	1	1.25%	1.25%
-0.41984 - -0.34833	-0.38409	1	1.25%	2.50%
-0.34833 - -0.27681	-0.31257	0	0.00%	2.50%
-0.27681 - -0.20530	-0.24106	3	3.75%	6.25% *
-0.20530 - -0.13379	-0.16954	1	1.25%	7.50%
-0.13379 - -0.062272	-0.098029	12	15.00%	22.50% *****
-0.062272 - 0.0092427	-0.026514	6	7.50%	30.00% **
0.0092427 - 0.080757	0.045000	10	12.50%	42.50% ****
0.080757 - 0.15227	0.11651	7	8.75%	51.25% ***
0.15227 - 0.22379	0.18803	12	15.00%	66.25% *****
0.22379 - 0.29530	0.25954	10	12.50%	78.75% ****
0.29530 - 0.36681	0.33106	10	12.50%	91.25% ****
0.36681 - 0.43833	0.40257	4	5.00%	96.25% *
0.43833 - 0.50984	0.47409	1	1.25%	97.50%
>= 0.50984	0.54560	2	2.50%	100.00%

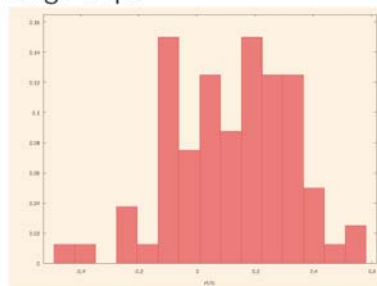
Missing observations = 1 (1.23%)

Graphical Interpretation: The Histogram

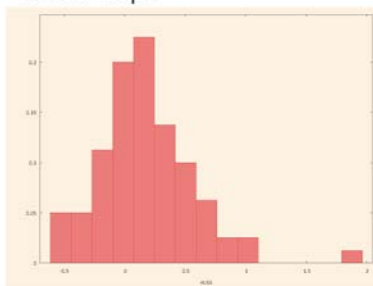


Comparing Distributions

Large Caps



Small Caps



Describing Distributions: Quantiles

- cdf \rightarrow How large a fraction f of observations are smaller or equal than a given number x ?
 - f is also called the (cumulative) frequency.

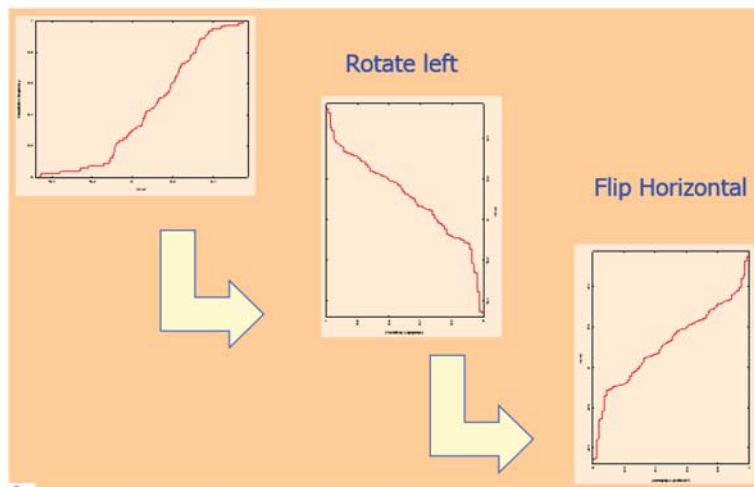
We can turn this question upside down:

- What is the smallest number x that is larger than a given proportion f of the data?
 - x is called the quantile at proportion f or f -th quantile Q_f .
- Special names:
 - Quartiles (4).
 - Quintiles (5).
 - Deciles (10).
 - Percentiles (100).
 - Median (2).

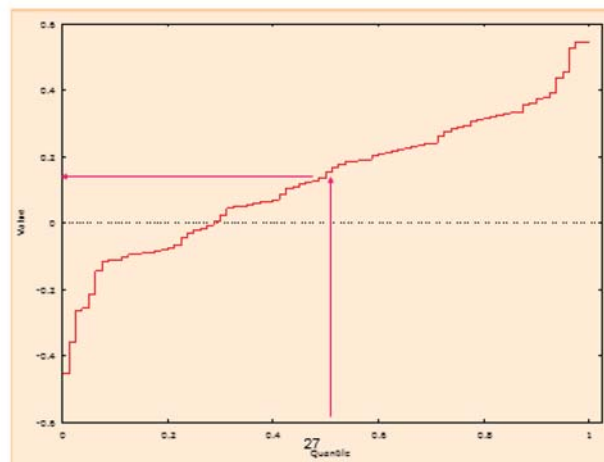
Example: Median US Large Caps

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Turning it Upside Down



Graphical Representation: Quantile Function



The Quantiles of US Asset Returns

	World Equity	US Large Stock	US Small Stocks	World Bond	US Bonds	US T- Bills	US Inflation
min	-39.9	-45.6	-52.7	-13.5	-8.7	-0.1	-10.3
1%	-28.5	-38.0	-51.2	-12.6	-7.7	0.0	-9.5
5%	-17.7	-22.3	-40.8	-5.1	-5.4	0.1	-2.3
10%	-15.9	-11.1	-22.1	-3.3	-3.6	0.3	-0.8
25%	-0.7	-2.5	-5.8	0.5	0.0	1.1	1.4
50%	13.6	13.6	18.1	4.1	4.9	3.2	2.9
75%	23.0	27.9	34.7	9.9	8.6	5.4	4.5
90%	29.1	36.0	65.5	18.6	15.6	7.7	8.8
99%	52.9	53.0	121.1	31.3	31.9	12.2	14.3
max	70.8	54.6	187.8	34.1	32.7	14.9	18.1

Summarizing a Distribution: Moments

- Like quantiles, moments also tell something about the location, scale, shape, etc. of the distribution.
- For instance, the mean can be interpreted as the “typical” outcome.
- Examples (compare with median):
 - Outcome of rolling a die.
 - Outcome of rolling two dice.

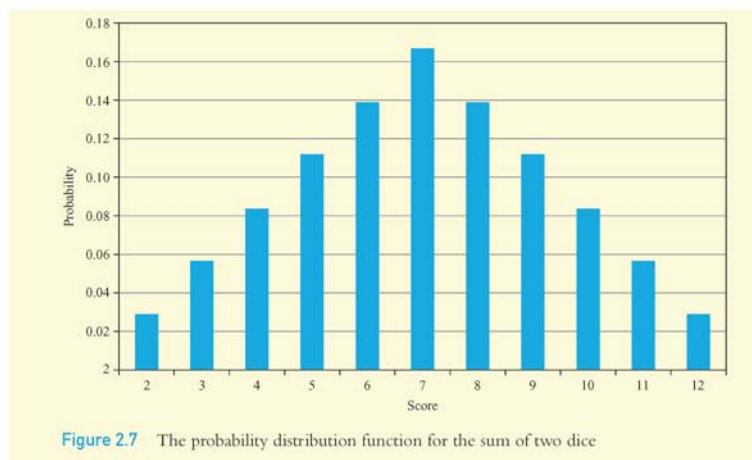
Summarizing a Distribution: Moments

- Like quantiles, moments also tell something about the location, scale, shape, etc. of the distribution.
- For instance, the mean can be interpreted as the “typical” outcome.
- Examples (compare with median):
 - Outcome of rolling a die.
 - Outcome of rolling two dice.
- Definition:

$$\bar{X} = \frac{1}{N} \sum_{i=1}^N x_i$$

$$\bar{X} = \sum_{i=1}^N x_i \cdot p(x_i)$$

Pdf of Rolling Two Dice



Triangle distribution

Expected Value

- We can generalize this definition, using the expectations operator:

$$\mathbb{E}(g(X)) = \sum_{i=1}^N g(x_i) p(x_i)$$

- Mean = **first moment** = $\mathbb{E}(X)$.
- Properties:
 - $\mathbb{E}(a) = a$
 - Linearity:
 - $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$
 - $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$
- Mean as fulcrum: $\mathbb{E}(X - \mathbb{E}(X)) = 0$

Be Careful (1)

- The mean (and expectations in general) are sensitive to outliers.
- Compare mean and median of:
 - 1, 3, 5
 - 1, 3, 14
- Alternatives:
 - Median.
 - Mode.
 - Trimmed mean.
 - Winsorized mean.

Application: LIBOR (London Interbank Offered Rate)

Barclays-Bank-plc	2.15	}	bbalibor Rate =
Bank-of-Tokyo-Mitsubishi-UFJ-Ltd	2.15		
HSBC	2.12		
Royal-Bank-of-Scotland-Group	2.11		
UBS AG	2.105		
Abbey National	2.1		
Bank of America	2.1		
Citibank NA	2.1		
Mizuho Corporate Bank	2.1		
Rabobank	2.1		
Royal Bank of Canada	2.1	}	<u>2.10063</u>
WestLB AG	2.1		
BNP-Paribas	2.05		
Lloyds-Banking-Group	2		
Deutsche-Bank-AG	1.95		
JP-Morgan-Chase	1.95		

Discard top and bottom quartiles and compute arithmetic average of the remaining 50% of the observations

<http://www.bba.org.uk/bba/jsp/polopoly.jsp?d=145&a=13777&artpage=4>

Be Careful (2)

- Linear functions are special:

$$\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$$

- This is not always the case:

$$\mathbb{E}(X^2) \neq [\mathbb{E}(X)]^2$$

$$\mathbb{E}\left(\frac{1}{X}\right) \neq \frac{1}{\mathbb{E}(X)}$$

- In general:

$$\mathbb{E}(g(X)) \neq g(\mathbb{E}(X))$$

Second (Central) Moment

- We want to tell something about the **dispersion** of the distribution
- $\mathbb{E}[X - \mathbb{E}(X)]^2$
- **Variance**: $\text{var}(X) = \mathbb{E}[X - \mathbb{E}(X)]^2$
- **Shortcut**: $\text{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$
- NOT a linear function!
- **Standard deviation** or **volatility** : $\sigma = \text{std}(x) = \sqrt{\text{var}(X)}$

Measures of Dispersion

- Range = maximum – minimum
- Mean Absolute Deviation: $\text{MAD}(X) = \mathbb{E} |X - \mathbb{E}(X)|$
- Mean Squared Deviation or Variance: $\text{var}(X) = \mathbb{E} [X - \mathbb{E}(X)]^2$
- Standard deviation: $\text{std}(X) = \sqrt{\text{var}(X)}$
- Coefficient of variation: $\text{CV} = \text{std}(X)/\mathbb{E}(X)$
- Interquartile range: $\text{IQR}(X) = Q_{0.75} - Q_{0.25}$

Properties of Variance

Adding a constant:

- Let $Y_i = a + X_i$ then $\text{var}(Y) = \text{var}(X)$

Multiplying by a constant:

- Let $Y_i = bX_i$ then $\text{var}(Y) = b^2 \text{var}(X)$

Standardization:

- Let

$$Y_i = \frac{X_i - \mathbb{E}(X)}{\text{std}(X)}$$

then $\mathbb{E}(Y) = 0$ and $\text{var}(Y) = 1$.

Chebyshev's Inequality

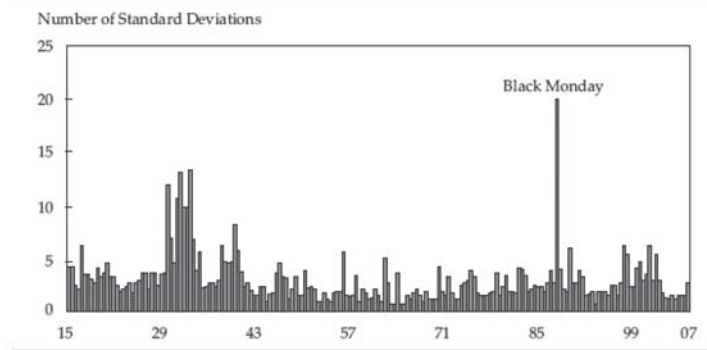
- Standard deviation is strongly linked to dispersion.
- Regardless of the distribution Chebyshev's inequality tells us:

$$\mathbb{P}[|X - \mathbb{E}(X)| \leq k \times \text{std}(X)] \geq 1 - \frac{1}{k^2}$$

k	proportion
1.25	36%
1.5	56%
2	75%
2.5	84%
3	89%
4	94%

Standardized Data ...

Figure 1. Daily Changes in the Dow, 1915–2007



Bogle, J.C., 2008, "Black Monday and Black Swans," *Financial Analysts Journal* 64(2), 30-40.

Black Swans or Black Turkeys?

Table 1. Black Turkeys All Over the Place

Asset Class	Period	Peak-to-Trough Decline
U.S. stocks (real total return)	1911–1920	51%
U.S. stocks (DJIA, daily)	1929–1932	89
Long U.S. Treasury bonds (real total return)	1941–1981	67
U.S. stocks	1973–1974	49
U.K. stocks (real total return)	1972–1974	74
Gold	1980–1985	62
Oil	1980–1986	71
Japanese stocks	1990–2009	82
U.S. stocks (S&P 500)	2000–2002	49
U.S. stocks (NASDAQ)	2000–2002	78
U.S. stocks (S&P 500)	2007–2009	57

Note: All returns are nominal price returns unless otherwise specified.

Siegel, L.B., 2010, Black Swan or Black Turkey? The state of economic knowledge and the crash of 2007–2009, *Financial Analysts Journal* 66(4), 6–10.

Higher Central Moments

- **Skewness and kurtosis:**

$$\text{skew}(X) = \frac{\mathbb{E}(X - \mathbb{E}(X))^3}{\text{std}(X)^3}$$

$$\text{kurt}(X) = \frac{\mathbb{E}(X - \mathbb{E}(X))^4}{\text{std}(X)^4}$$

- For **normal distributions**, we have:
 - skewness = 0.
 - kurtosis = 3.
- Excel and gretl compute **excess kurtosis** = kurtosis - 3.

Sum Stats US Asset Returns in Excel

	World Equity	US Large Stock	US Small Stocks	World Bond	US Bonds	US T- Bills	US Inflation
mean	11.46	12.15	17.95	6.14	5.68	3.75	3.13
standard deviation	18.57	20.26	38.71	9.09	8.09	3.15	4.29
skewness	-0.03	-0.36	1.18	0.88	0.99	1.03	0.23
kurtosis	0.79	-0.07	3.76	1.16	1.63	1.10	3.05

“Fat Tails”

Outline

- 1 Time Value of Money
- 2 Valuing Bonds and Stocks
- 3 Computing Returns and Return Indices
- 4 Descriptive Statistics and Frequency Distributions
- 5 Theoretical Distributions and Hypothesis Testing**
- 6 Portfolio Returns and Risk
- 7 Portfolio Selection
- 8 Univariate Regression
- 9 CAPM and the Single Index Model
- 10 Forwards and Futures
- 11 Introduction to Options

Theoretical Return Distributions

- Using historical distributions as a basis for decision taking is fine but...
 - Do they capture all possibilities?
 - i.e. are they representative for the future? T
 - They are somewhat "clumsy" to work with.
- Summarizing them using descriptives is fine but...
 - Which descriptives to use?
 - Are the descriptives sufficiently "stable"/accurate/representative?
- Theoretical distributions may be useful.

Examples

- Bernoulli.
- Binomial.
- Discrete uniform distribution.
- Continuous uniform distribution.
- Normal distribution.

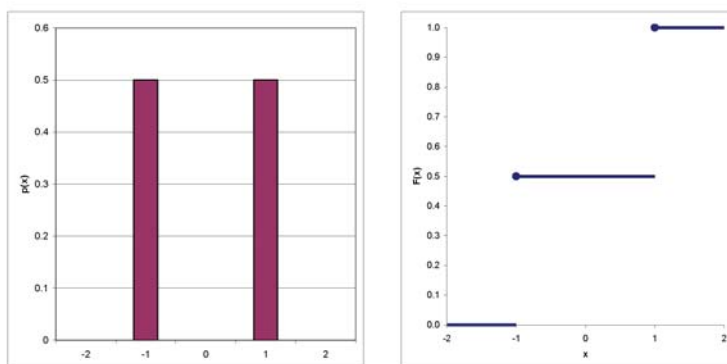
Bernoulli

- A distribution used to describe a RV that takes **two** possible outcomes:
 - Success or failure.
 - Odd or even.
 - Positive or negative return.
 - Head or tails.
 - Boy or girl.
 - Default or not.
 - Zero or one...
- Only one parameter defines this distribution: probability of success, q (or probability of failure).
- The two outcomes are often given the (arbitrary) values 0 and 1.
- Hence,

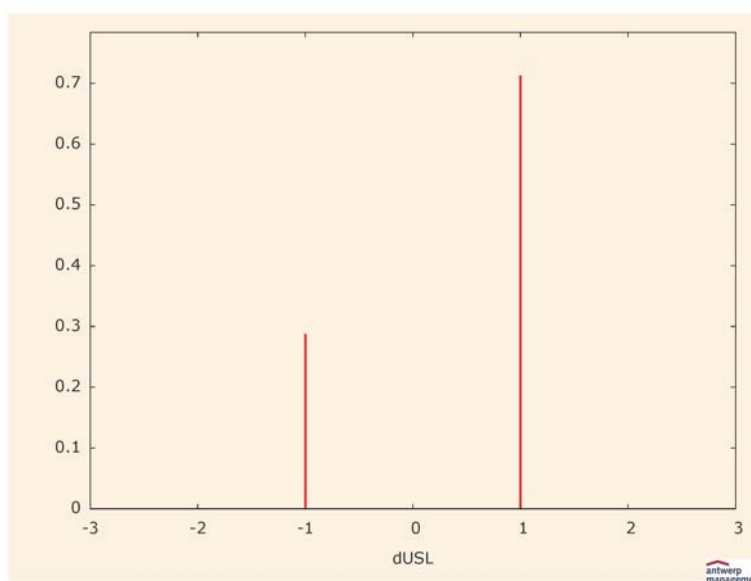
$$p(0) = q, p(1) = 1 - q$$

Bernoulli Example

- Here, we follow another convention for the two numerical values.
- Negative (-1) or positive (+1) returns, $q = 50\%$

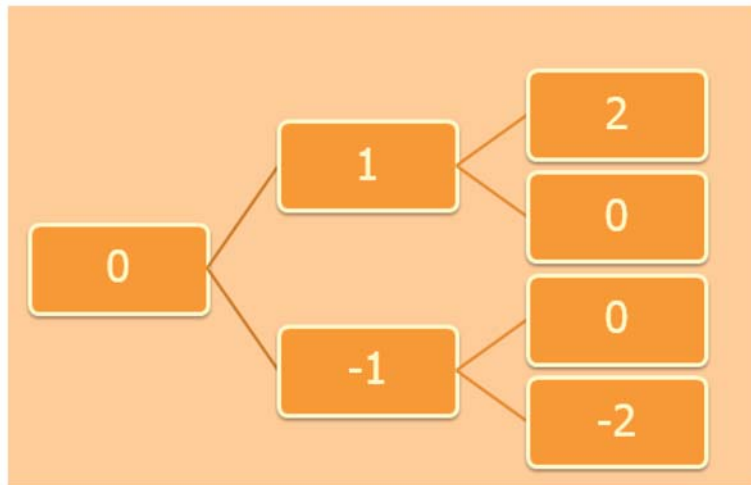


Annual US Large Cap Returns



Binomial Distribution

- Sum of (i.i.d.) Bernoulli distributed RVs.



- Remark: the notion of i.i.d.

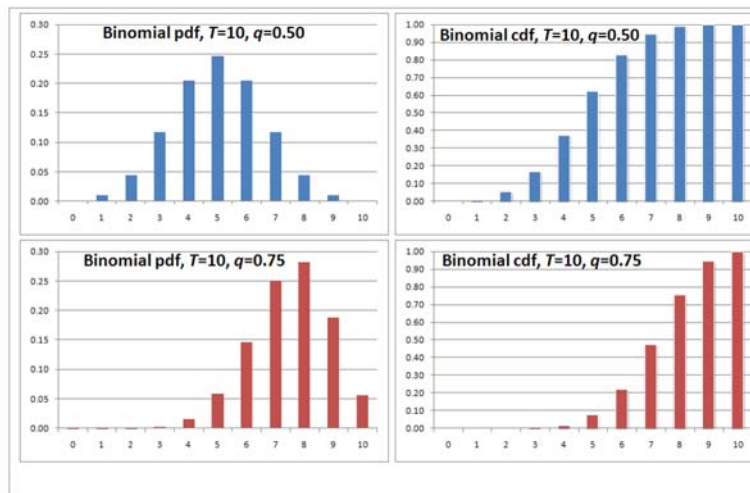
Binomial Distribution

- What do we need to specify this distribution?
 - Probability of "success" (q).
 - Number of RVs to sum (number of "trials", T).
- The pdf then looks like:

$$\begin{cases} p(x) = \frac{T!}{x!(T-x)!} q^x (1-q)^{T-x}, & x \in \{0, 1, 2, \dots, T\} \\ p(x) = 0 & \text{otherwise} \end{cases}$$

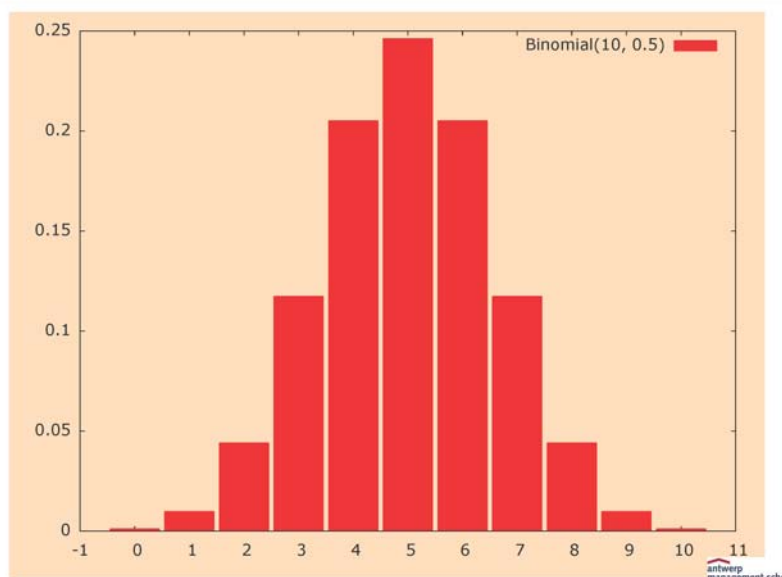
- Let's try it out ($T=2$, $q=50\%$):
 - $p(2) = 2!/(2!0!) \times 0.50^2 \times 0.50^0 = 0.25$ (remember $0!=1$ by convention)
 - $p(1) = 2!/(1!1!) \times 0.50^1 \times 0.50^1 = 2 \times 0.25 = 0.50$
 - $p(0) = 2!/(0!2!) \times 0.50^0 \times 0.50^2 = 0.25$

Illustration Binomial Distribution

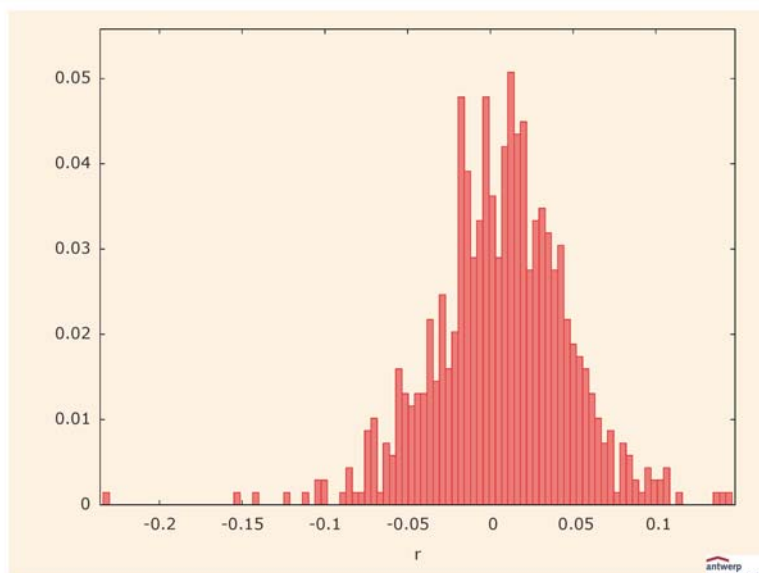


- In Excel: `BINOMDIST(x,T,q,false/true)`

In gretl (after some manipulations)

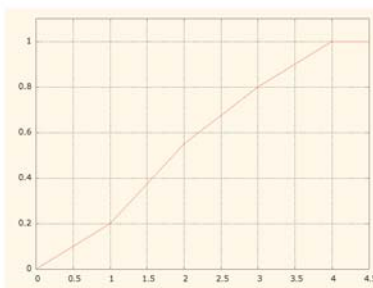
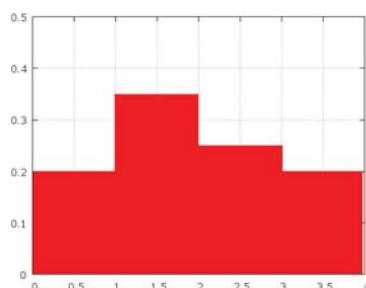


Histogram



Continuous Distributions

- When there is an **uncountable** number of outcomes, the “traditional” pdf breaks down
- Let's take a simple example:



Example and Intuition

- In the histogram, probabilities are spread uniformly across the intervals:
 - $\mathbb{P}([1, 2]) = 0.35 \times (2 - 1) = 35\% = F(2) - F(1) = 0.55 - 0.2$
 - $\mathbb{P}([1, 1.5]) = 0.35 \times (1.5 - 1) = 17.5\% = F(1.5) - F(1) = 0.375 - 0.2$
 - $\mathbb{P}([1, 1.1]) = 0.35 \times (1.1 - 1) = 3.5\% = F(1.1) - F(1) = 0.235 - 0.2$
- So probabilities are **surfaces**: height of bar \times length interval.
 - $\mathbb{P}([1, 1 + \Delta x]) = 0.35 \times \Delta x = F(x + \Delta x) - F(x)$

$$\Rightarrow 0.35 = \frac{F(x + \Delta x) - F(x)}{\Delta x}$$

- Let's look at limiting cases:

$$\lim_{\Delta x \rightarrow 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \frac{dF(x)}{dx} = f(x)$$

- "Rate of probability".

Probability Density Function

- In our example $f(x)$ is piecewise constant, but it can be a continuous function.
- The function $f(x)$ is called the **probability density function** (pdf again).
- It measures the rate at which the probability increases when we move further in the interval.
- As probabilities are surfaces, we have to integrate the pdf:

$$dF(x) = f(x)dx$$

$$\mathbb{P}([x_1, x_2]) = F(x_2) - F(x_1) = \int_{x_1}^{x_2} dF(x) = \int_{x_1}^{x_2} f(x)dx$$

Properties Pdf

- As $F(x)$ is non-decreasing, $f(x)$ is **non-negative** (compare with $p(x)$).
- The pdf integrates to **one** (compare with $p(x)$ summing to one):

$$\int_{-\infty}^{+\infty} f(x) dx = 1$$

Note that $f(x) > 1$ is possible (compare with $p(x)$).

- The pdf is the **derivative** of the cdf.

$$f(x) = \frac{dF(x)}{dx}$$

Properties Pdf (Cont'd)

- The cdf is the **integral** of the pdf:

$$F(x) = \int_{-\infty}^x f(u) du$$

- The probability of **one single** outcome is **zero**!

$$\mathbb{P}(\{x_0\}) = \int_{x_0}^{x_0} f(x) dx = 0$$

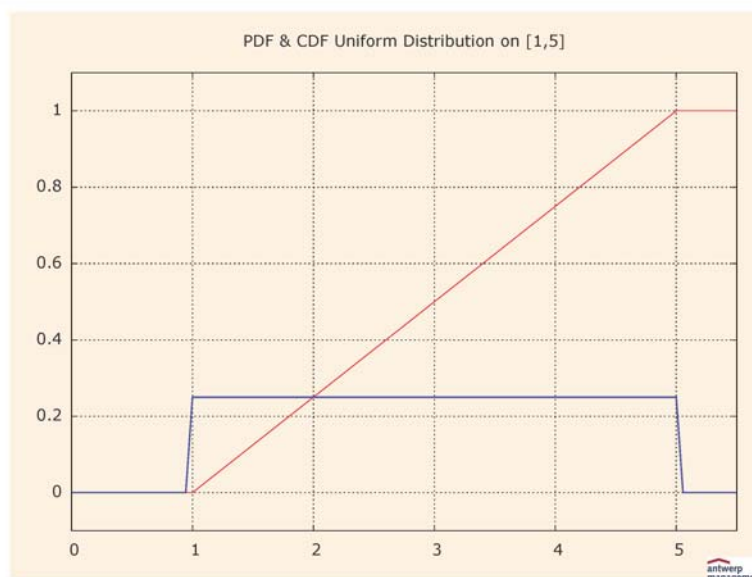
Example: Continuous Uniform Distribution

- The uniform distribution on an interval $[a, b]$ has pdf $f(x) = 1/(b - a)$ for $x \in [a, b]$ and 0 else.
- Why? It is constant and integrates to 1:

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x) dx &= \int_a^b \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b dx = \frac{1}{b-a} x \Big|_{x=a}^{x=b} \\ &= \frac{1}{b-a} (b-a) = 1 \end{aligned}$$

- Its cdf is:
 - Zero below a .
 - $(x - a)/(b - a)$ for $a \leq x \leq b$.
 - One above b
- The probability of an outcome in a subinterval $[c, d] = (d - c)/(b - a)$ (Check it!).

Example



Summarizing a Distribution: Moments

- Expectations operator:

$$\begin{aligned}\mathbb{E}(g(X)) &= \int_{-\infty}^{+\infty} g(x) \cdot f(x) dx \\ &= \sum_i g(x_i) \cdot p(x_i)\end{aligned}$$

- First moment, mean:

$$\begin{aligned}\mu = \mathbb{E}(X) &= \int_{-\infty}^{+\infty} x \cdot f(x) dx \\ &= \sum_i x_i \cdot p(x_i)\end{aligned}$$

Do you recognize the arithmetic mean?

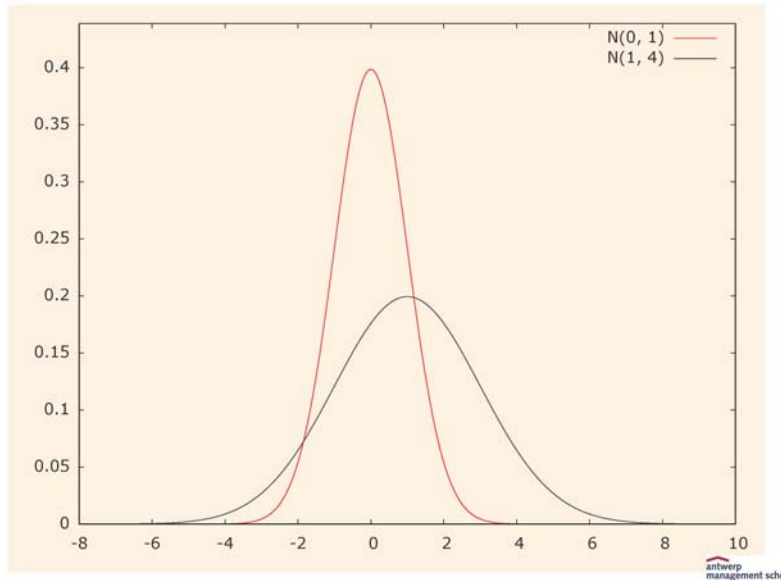
Examples

- Continuous uniform distribution function on $[a, b]$.
- Mean:

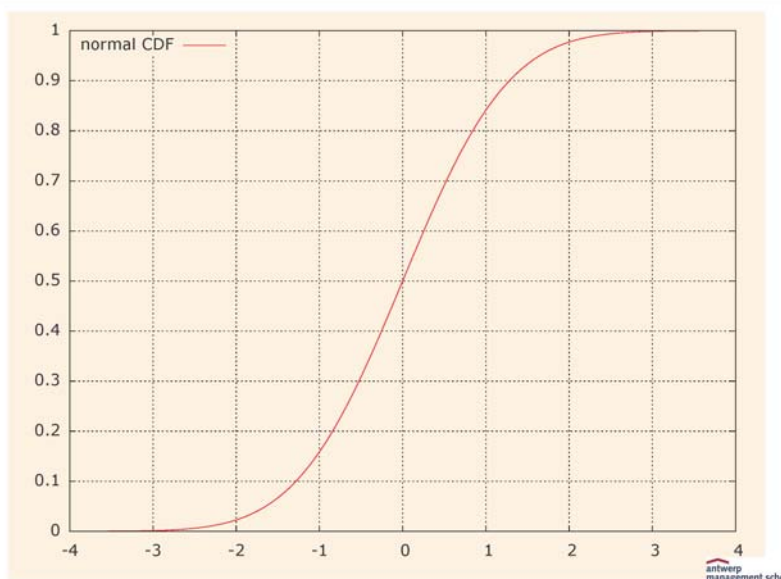
$$\begin{aligned}\mu &= \mathbb{E}(X) = \int_{-\infty}^{+\infty} x \cdot f(x) dx \\ &= \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b \\ &= \frac{1}{2(b-a)} (b^2 - a^2) = \frac{b+a}{2}\end{aligned}$$

- Compute the variance for $a = 0$ and $b = 1 \dots$

The Normal Distribution (Gaussian Distribution)



The cdf of the Normal Distribution



Properties

- Completely symmetric around its location parameter μ .
- The parameter σ determines the scale or width of the distribution. (note: in the gretl graph σ^2 is reported)
- A standard normal distribution has $\mu = 0$ and $\sigma = 1$.
- A general normal RV x can be transformed into a standard normal RV z as follows: $z = (x - \mu)/\sigma$.
- The pdf is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

- The cdf does not exist in closed form. See tables or Excel or ...
- In Excel you have the `normdist()` function, both for the cdf and the pdf.

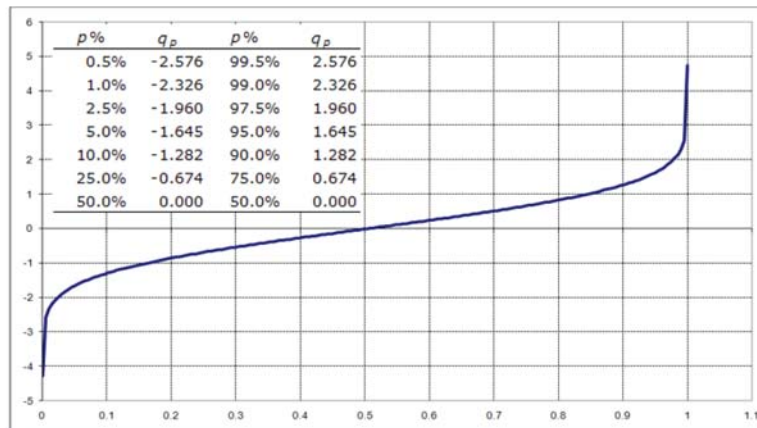
Check It Out

Normal distribution

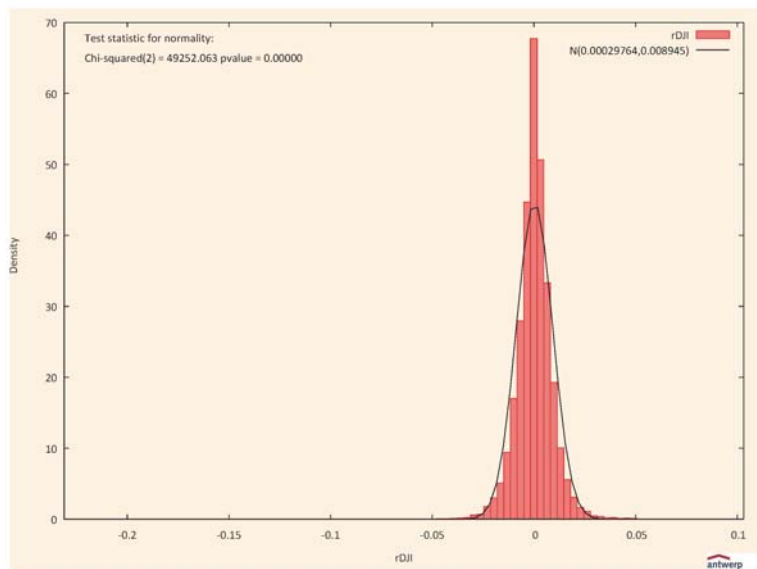
Chebyshev

$\mathbb{P}([\mu - \sigma, \mu + \sigma]) = 68.27\%$	$\mathbb{P}([\mu - \sigma, \mu + \sigma]) \geq 0\%$
$\mathbb{P}([\mu - 2\sigma, \mu + 2\sigma]) = 95.45\%$	$\mathbb{P}([\mu - 2\sigma, \mu + 2\sigma]) \geq 75\%$
$\mathbb{P}([\mu - 3\sigma, \mu + 3\sigma]) = 99.73\%$	$\mathbb{P}([\mu - 3\sigma, \mu + 3\sigma]) \geq 89\%$
$\mathbb{P}([\mu - 4\sigma, \mu + 4\sigma]) = 99.99\%$	$\mathbb{P}([\mu - 4\sigma, \mu + 4\sigma]) \geq 94\%$

The Quantiles of the Normal Distribution



So, Are (Daily DJIA) Return Distributions Normal?

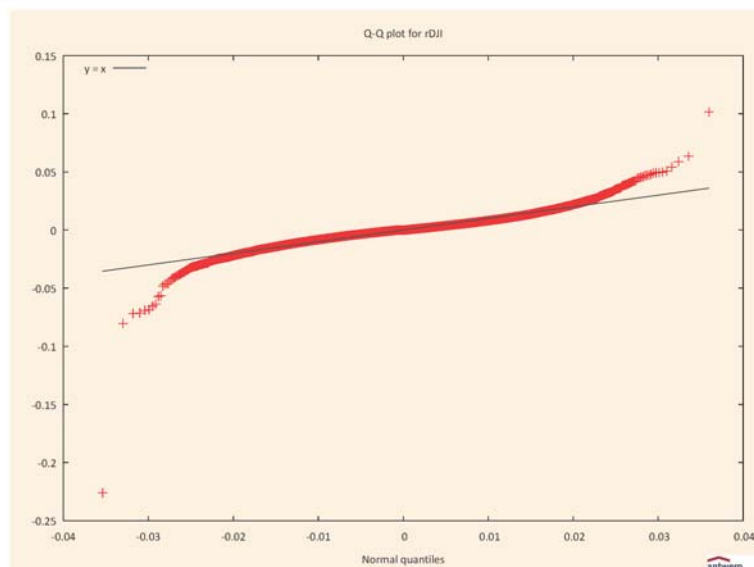


Sum Stats

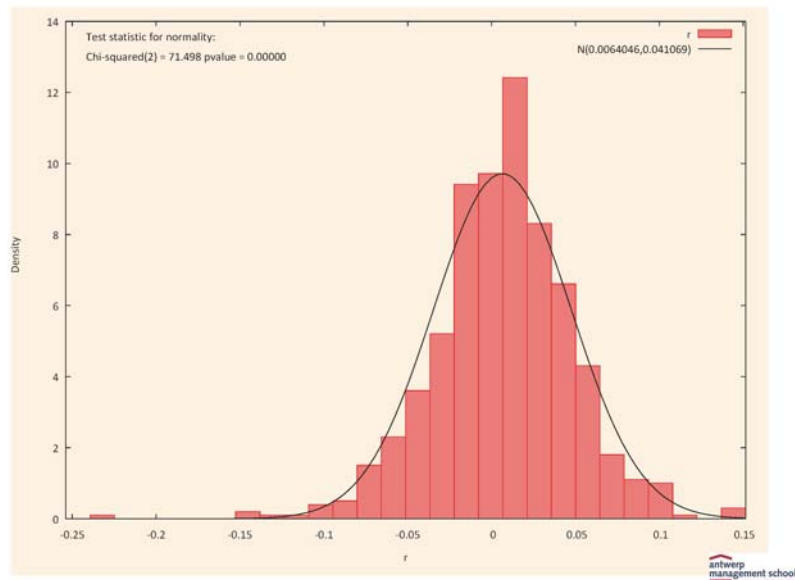
Summary statistics, using the observations 1951/01/02 - 2008/08/27
for the variable 'rDJI' (15041 valid observations)

Mean	0.00029764
Median	6.5072e-005
Minimum	-0.22611
Maximum	0.10149
Standard deviation	0.0089450
C.V.	30.053
Skewness	-1.0649
Ex. kurtosis	32.749

Quantile-Quantile Plot



And Monthly Data?

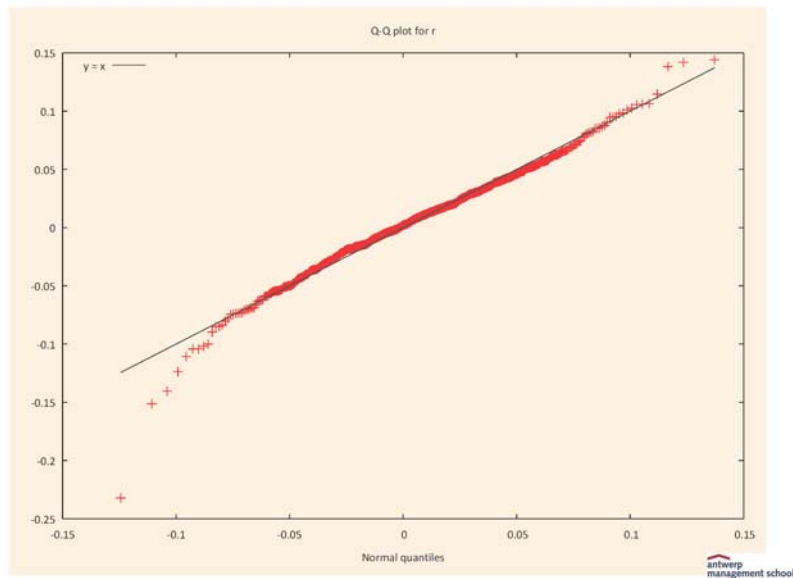


Sum Stats

Summary statistics, using the observations 1951:01 - 2008:07
for the variable 'r' (690 valid observations)

Mean	0.0064046
Median	0.0084571
Minimum	-0.23216
Maximum	0.14414
Standard deviation	0.041069
C.V.	6.4125
Skewness	-0.38844
Ex. kurtosis	2.1471

QQ Plot Monthly



Other Candidate Distributions?

Student's t distribution

- Pdf:

$$f_X(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \sqrt{\pi n}} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}$$

- Note the parameter n ('degree of freedom')
- Symmetric and fat-tails

The Lognormal Distribution

- X is lognormally distributed iff $\ln(X)$ is normally distributed.
- Let $Y = \ln(X)$ normally distributed with $\mathbb{E}(Y) = m$ and $\text{std}(Y) = s$
- Then:
 - $X = \exp(Y)$ varies between 0 and $+\infty$
 - X is **lognormally distributed**.
 - pdf:

$$f(x) = \frac{1}{\sqrt{2\pi}sx} \exp\left(-0.5\left(\frac{\ln x - m}{s}\right)^2\right), \quad 0 < x < \infty$$

$$= 0, \quad \text{otherwise}$$

- $\mathbb{E}(X) = \exp(m + s^2/2)$ (check Jensen's inequality)
- $\text{var}(X) = \exp(2m + s^2)(\exp(s^2) - 1)$.

Application: Long-Term Returns

- What is the link between return distributions when returns are measured over increasingly longer period?
- Answer: Complicated in general.
- Easy when:
 - Returns are lognormally distributed.
 - When they are independent.
 - When they are identically distributed

Covariance and Correlation

- Expectations are linear and reduce to univariate expectations:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

- Variance is **NOT** linear:

$$\text{var}(aX + bY) = a^2\text{var}(X) + b^2\text{var}(Y) + 2ab\text{cov}(X, Y)$$

- It involves a **covariance** term:

$$\begin{aligned}\text{cov}(X, Y) &= \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) \\ &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)\end{aligned}$$

- **Correlation** is a **standardized** covariance:

$$\rho = \text{cor}(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}$$

Covariance and Independence

- When RVs are independent, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

- The covariance can be written as:

$$\text{cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

- Therefore when RVs are independent, then $\text{cov}(X, Y) = 0$.

- Independence **implies** zero correlation.

- But **not** the other way around:

RVs can have zero correlation and still be dependent!

Distribution of the Arithmetic Mean

- What about the distribution of the arithmetic mean?
- The arithmetic mean is a sum of iid RVs.
- Unfortunately, the sum of iid RV generally has NOT the same distribution.
 - Remember the example of the sum of Bernoulli RVs.
- When the data are normally distributed, the sum is also normally distributed.
- For most other distributions, the distribution of the sum is not known.
- Fortunately, in most cases that unknown distribution “looks” more and more like the normal distribution as the sample size increases.

→ Asymptotical normal distribution

Example: Uniform RVs

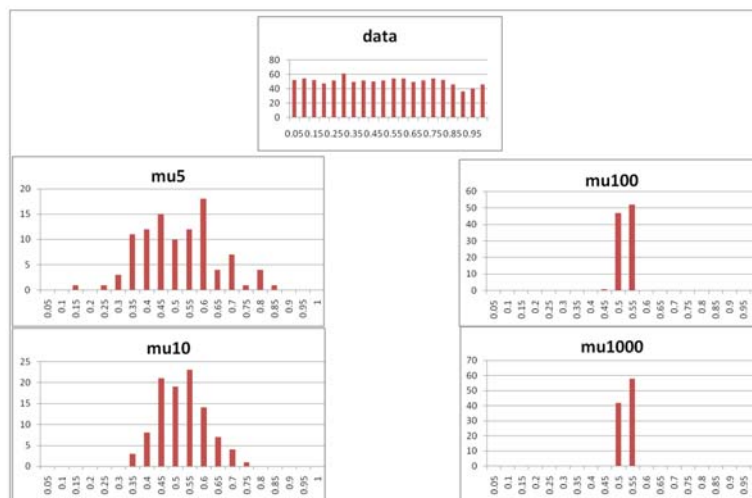
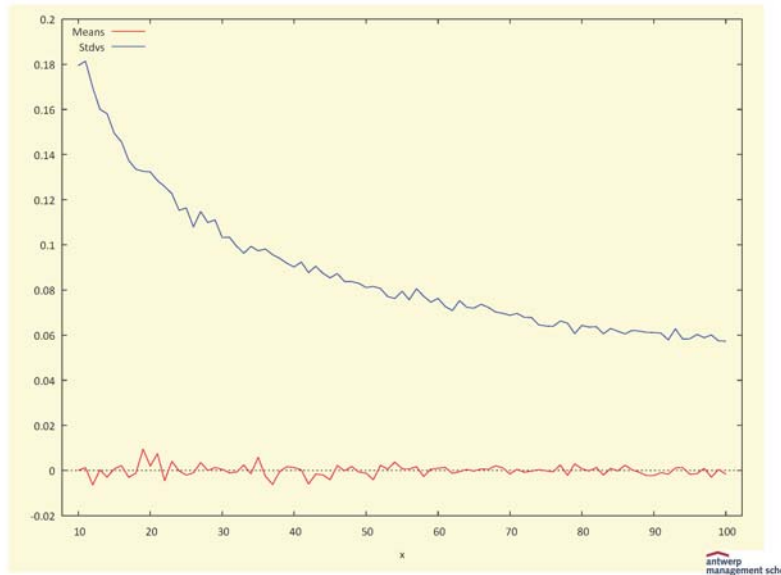


Illustration Consistency



Variance of the Mean

- So, the arithmetic average is 'approximately' normally distributed.
- It is unbiased, so its mean is the true μ .
- But what is its variance?

$$\begin{aligned}
 \text{var}(\bar{R}) &= \text{var}\left(\frac{1}{T} \sum_{t=1}^T R_t\right) \stackrel{\text{iid}}{=} \frac{1}{T^2} \sum_{t=1}^T \text{var}(R_t) \\
 &= \frac{1}{T^2} \sum_{t=1}^T \text{var}(\mu + e_t) = \frac{1}{T^2} \sum_{t=1}^T \text{var}(e_t) \\
 &= \frac{T}{T^2} \sigma^2 = \frac{\sigma^2}{T}
 \end{aligned}$$

Note: goes to zero when the sample size T increases (cfr. consistency)

Making Statements about the Mean

- ① Confidence intervals.
- ② Hypothesis testing.

Making Statements about the Mean

Confidence intervals:

- Within which interval does the real (population) mean lie, given that we know the arithmetic mean?
- With a given confidence level.
- Is usually two-sided.
- Asymptotic normality \rightarrow take a number of standard deviations from the mean.

Confidence Interval

- Let confidence level = 95% and symmetric.
- We look for the lower bound m and upper bound M for which:

$$\mathbb{P}[\bar{R} \leq m] = 2.5\% \text{ and } \mathbb{P}[\bar{R} \geq M] = 2.5\% \Rightarrow \mathbb{P}[m \leq \bar{R} \leq M] = 95\%$$

- As the arithmetic mean is (asymptotically) normally distributed, we know:
 - $m = \text{mean} - 1.96 \times \text{std} = \mu - 1.96 \times \sigma / \sqrt{T}$
 - $M = \text{mean} + 1.96 \times \text{std} = \mu + 1.96 \times \sigma / \sqrt{T}$
- We can now rewrite the interval.

Confidence Interval for μ

$$\begin{aligned} \Pr[m \leq \bar{R} \leq M] &= 95\% \\ \Pr\left[\mu - 1.96 \frac{\sigma}{\sqrt{T}} \leq \bar{R} \leq \mu + 1.96 \frac{\sigma}{\sqrt{T}}\right] &= 95\% \\ \Pr\left[-\bar{R} - 1.96 \frac{\sigma}{\sqrt{T}} \leq -\mu \leq -\bar{R} + 1.96 \frac{\sigma}{\sqrt{T}}\right] &= 95\% \\ \Pr\left[\bar{R} + 1.96 \frac{\sigma}{\sqrt{T}} \geq \mu \geq \bar{R} - 1.96 \frac{\sigma}{\sqrt{T}}\right] &= 95\% \\ \Pr\left[\bar{R} - 1.96 \frac{\sigma}{\sqrt{T}} \leq \mu \leq \bar{R} + 1.96 \frac{\sigma}{\sqrt{T}}\right] &= 95\% \end{aligned}$$

- Interpretation: in 95% of the cases the confidence interval will contain the true population value μ .
- But what is σ ?

What about σ ?

- In practice, the standard deviation is not known. It has to be estimated.
- The following sample estimators could be used:

$$\frac{1}{T} \sum_{t=1}^T (R_t - \bar{R})^2 \quad \text{or} \quad \frac{1}{T-1} \sum_{t=1}^T (R_t - \bar{R})^2$$

- When T is very large, the difference between both is negligible.
- In Excel and gretl, the second alternative is implemented.
- Note: using the sample estimator changes the distribution somewhat, but again in large samples these differences can be ignored.

Example

- 95% interval for expected return US Large caps
- Average return and standard deviation for 1926-2009 resp.: 11.630% and 20.561%.

Lower bound:

$$11.63 - 1.96 \frac{20.561}{\sqrt{84}} = 7.233$$

Upper bound:

$$11.63 + 1.96 \frac{20.561}{\sqrt{84}} = 16.027$$

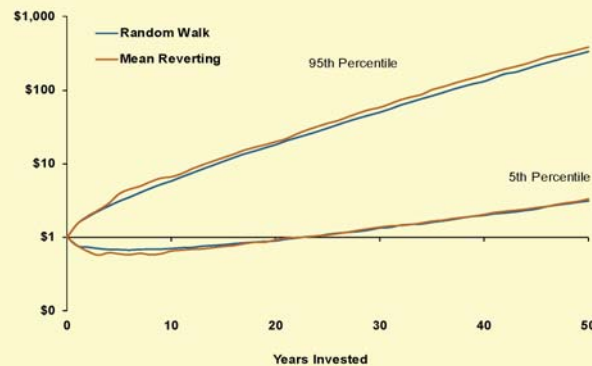
Confidence interval:

$$\Pr[7.233 \leq \mu \leq 16.027] = 95\%$$

Are Stocks Less Risky in the Long Run?

Are Stocks Less Risky in the Long Run?

Bootstrapped distribution of real growth of wealth in World ex US equities



Source: Lee, M., 2012, A Century of Global Returns, Dimensional Fund Advisors. Based on data by Dimson, Marsh & Staunton (DMS).

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Hypothesis Tests

- Specify a null hypothesis.
- Specify an alternative hypothesis.
 - One-sided.
 - Two-sided.
- Specify a statistic.
- Determine the distribution of the statistic under the null hypothesis.
- Specify a confidence level or a significance level.
- Determine the plausibility of the null hypothesis.
- If it is not plausible, reject the null hypothesis.
- **Never accept a null hypothesis! You can only fail to reject a null hypothesis.**

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Example: Arithmetic Mean

- H_0 : arithmetic mean (AM) = a given constant μ_0 .
The alternative can be: "is larger", "is smaller", "is not".
Let's take H_a : arithmetic mean is smaller than μ_0 .
- Evidence against H_0 is then how likely it is that we find numbers as extreme or even more extreme than the actual AM, assuming H_0 is true.
- Extremeness depends on H_a . Here it is small values for z : $z = \frac{\bar{R} - \mu_0}{\sigma/\sqrt{T}}$
- So, compute the probability to get values less than z :

$$\Pr\left[Z \leq z = \frac{\bar{R} - \mu_0}{\sigma/\sqrt{T}}\right]$$

- If this probability is small, we reject H_0 . If not, we cannot reject H_0 .
- What is small? \rightarrow less than the significance level.
- When significance level = 5%, extreme values are 1.645 σ below the mean.
So, reject when

$$z = \frac{\bar{R} - \mu_0}{\sigma/\sqrt{T}} \leq -1.645$$

Example: Arithmetic Mean (Cont'd)

When H_a : arithmetic mean is larger than μ_0 , extreme values are large z .

- Compute probability $\Pr\left[Z \geq z = \frac{\bar{R} - \mu_0}{\sigma/\sqrt{T}}\right]$
- Reject when z is more than 1.645 (significance level 5%).

When H_a : arithmetic mean is not equal to μ_0 , extreme values are large absolute values of z .

- Compute probability $\Pr[Z \geq |z|]$
- Reject when $|z|$ is more than 1.96 (significance level 5%).

Errors

	H_0 correct	H_0 incorrect
H_0 rejected	Type I error	Correct
H_0 not rejected	Correct	Type II error

Two types of error:

- Type I error
 - is the probability to reject a true H_0 . This is controlled by setting the significance level. Typically, this is a small percentage, as we want to avoid convicting an innocent suspect.
 - Type I error = significance level of test.
- Type II error
 - is the probability not to reject a wrong H_0 . We try to choose a test that minimizes this probability. This is a powerful test.
 - Formally, the power of a test is the probability to reject a wrong H_0 . So, power = 1 – Type II error.

So, Why the Arithmetic Mean

We use the arithmetic mean because:

- It is unbiased;
- We know its (approximate) distribution.
- It is efficient.

For an unbiased estimator, efficiency reflects accuracy.

We could have used the median, but:

- It is only a valid estimator for μ when the distribution is symmetric.
- It is less efficient.

Summarizing

- t -test (or rather a z -test).
- Reject the null hypothesis.
- Retain the null hypothesis (don't accept!).
- Type I and Type II error.
- P-value or marginal significance level.
- Estimate standard deviation!

Accepting a Null Hypothesis

Assume $AM=0.5$, with standard error 0.1

Test the following null hypothesis:

- H_0 : arithmetic mean = 0 versus
- H_a : arithmetic mean > 0

Can you reject the null hypothesis?

Now test an alternative null hypothesis:

- H_0 : arithmetic mean = 1 versus
- H_a : arithmetic mean < 1

Can you reject this null hypothesis?

Conclusion: Accepting a null hypothesis doesn't make any sense

Outline

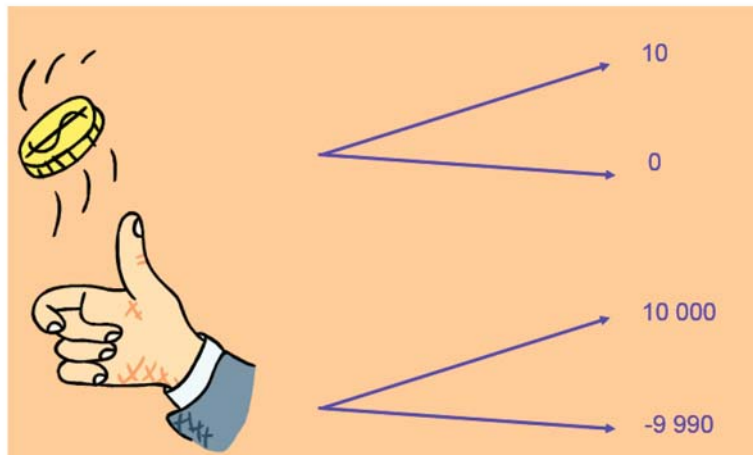
- 1 Time Value of Money
- 2 Valuing Bonds and Stocks
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How to Determine Asset Prices

Suggestion:

- Determine possible future values (incl. dividends, income, ...).
- Compute expected future value.
- Discount the expected future value at the appropriate interest rate.
- The interest rate accounts for time ('term to maturity').

But What about Risk Aversion?



St. Petersburg Paradox (Daniel Bernouilli, 1738)

The diagram shows four hands flipping coins, representing the sequence of outcomes in the St. Petersburg Paradox. Below the illustrations, the expected value calculation is shown:

$$\begin{aligned}
 & \frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{8} \times 4 + \frac{1}{16} \times 8 + \dots \\
 &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \\
 &= \text{INFINITY} = \text{price?}
 \end{aligned}$$

Expected outcome cannot be equal to price

How to Determine Asset Prices? Again

Solution:

- Determine possible future values (incl. dividends, income, ...).
- Compute expected future value.
- Discount the expected future value at the appropriate **discount** rate.
- The rate takes into account the term to maturity **and risk**.
- **Question:** How do we adjust for risk?

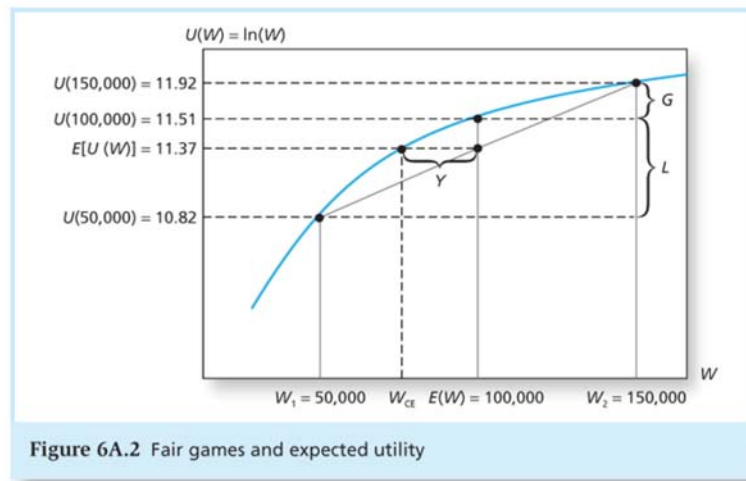
Solution: Utility Function

- Let's "weight" the possible outcomes and compute expected value.
- Cramer proposed $U(x) = \sqrt{x}$.
- St. Petersburg paradox:

$$\frac{1}{2}\sqrt{1} + \frac{1}{4}\sqrt{2} + \frac{1}{8}\sqrt{4} + \frac{1}{16}\sqrt{8} + \dots = \frac{1/2}{1 - \sqrt{2}/2} = 1.707.$$

- \Rightarrow Expected utility is 1.707.
- The break-even price is therefore: $1.707^2 = \text{€ } 2.914$.
- This is called the **Certainty Equivalent (CE)**

Fair Games versus Expected Utility

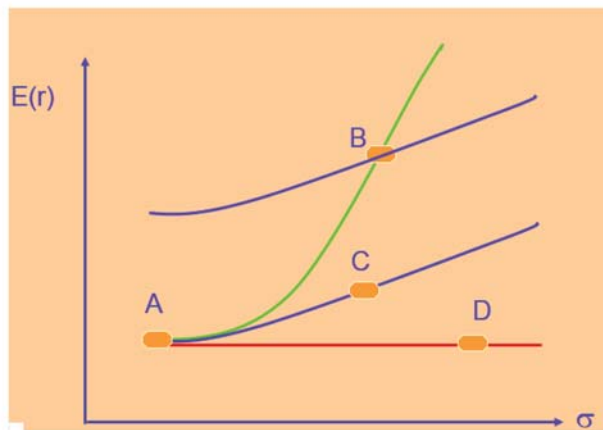


Mean-Variance Model

- Mean-variance model by Markowitz: a portfolio A is preferred to portfolio B if:
 - Expected return A is higher or equal than expected return B.
 - Variance return A is lower or equal than variance return B.
- Is consistent with a **MV-utility function**:

$$U = \mathbb{E}(r) - \frac{1}{2}A\sigma^2$$

Iso-utility Curves / Indifference Curves



When is Variance OK?

- Quadratic utility function.
- Normality returns.
- In general there is a preference for **skewness**, but
 - rapidly decreases in portfolios.
 - expected return and variance are good approximation.

Skewness May Be Important



Getting the Risk Aversion Parameter

Usually by means of questionnaires, with questions such as:

Example

Suppose that you are the only income earner in the family, and you have a good job guaranteed to give you your current (family) income every year of your life. You are given the opportunity to take a new and equally good job, with 50-50 chance that it will double your (family) income, and a 50-50 chance that it will cut your (family) income by a third. Would you take the new job? (Source: Barsky, Juster, Kimball & Shapiro (1997, p. 540))

Risk and Return

	Insurance A	Building B
Normal	40%	0%
Disaster	-10%	60%
$E(r)$	15%	30%
σ	25%	30%

Note: in this two-scenario-case with equal probabilities, the standard deviation is equal to the deviation of the outcomes from the expected return.

And Portfolios? Return

- (Expected) return is **linear**:

	Stock A	Stock B	Portfolio
Price t=0	100	50	150
Price t=1 (OK)	140	50	190
Price t=1 (Earthquake)	90	80	170
Expected Price	115	65	180
Expected return	15%	30%	20%

$$r_{A+B} = w_A r_A + w_B r_B$$

$$E(r_{A+B}) = w_A E(r_A) + w_B E(r_B)$$

And Risk?

- Variance is **NOT** linear:
 - $\text{Var}(A + B) = 1/2 \times (6.7\%)$
 - $\text{std}(A + B) = 6.7 (= 10/150, \text{ cf. note two slides ago})$
- Apparently, there is **risk reduction**: $6.7\% < 25\%$ and 30%

Return and Risk

- Expected Return

$$\mathbb{E}(r_{A+B}) = w_A \mathbb{E}(r_A) + w_B \mathbb{E}(r_B)$$

- Risk

$$\sigma_{A+B}^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2w_A w_B \text{COV}(r_A, r_B)$$

Capital Allocation

- Proportion safe - risky.
- Later:
 - Asset Allocation: filling in the risky part by asset classes
 - Security Selection: filling in the asset classes by individual assets

Risk and Return

Let

- f = risk-free asset
- p = risky asset
- c = combined portfolio

Expected return C:

$$\begin{aligned} E(r_c) &= w_f E(r_f) + w_p E(r_p) \\ &= w_f r_f + w_p E(r_p) \\ &= r_f + w_p [E(r_p) - r_f] \end{aligned}$$

Risk C:

$$\begin{aligned} \sigma_c^2 &= w_f^2 \sigma_f^2 + w_p^2 \sigma_p^2 \\ &\quad + 2w_f w_p \text{cov}(r_f, r_p) \\ &= w_p^2 \sigma_p^2 \\ \sigma_c &= |w_p| \sigma_p \end{aligned}$$

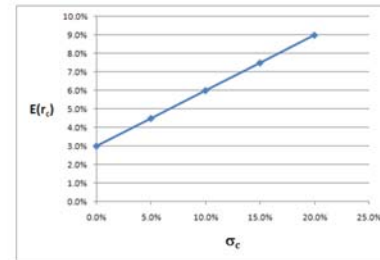
Example

$$E(r_c) = r_f + [E(r_p) - r_f]$$

$$\sigma_c = w_p \sigma_p$$

r_f	3%
$E(r_p) - r_f$	6%
σ_p	20%

w_p	$E(r_c)$	σ_c
0%	3.0%	0.0%
25%	4.5%	5.0%
50%	6.0%	10.0%
75%	7.5%	15.0%
100%	9.0%	20.0%



The Capital-Allocation Line (The Maths)

$$\begin{aligned} E(r_c) &= r_f + w_p [E(r_p) - r_f] \\ &= r_f + \frac{\sigma_c}{\sigma_p} [E(r_p) - r_f] \\ &= r_f + \frac{[E(r_p) - r_f]}{\sigma_p} \cdot \sigma_c \end{aligned}$$

In words:

Expected return = reward time + risk premium

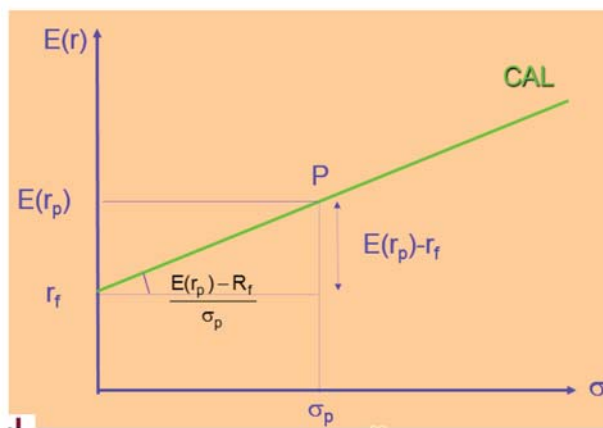
Risk premium = price of risk \times quantity of risk

Sharpe Ratio

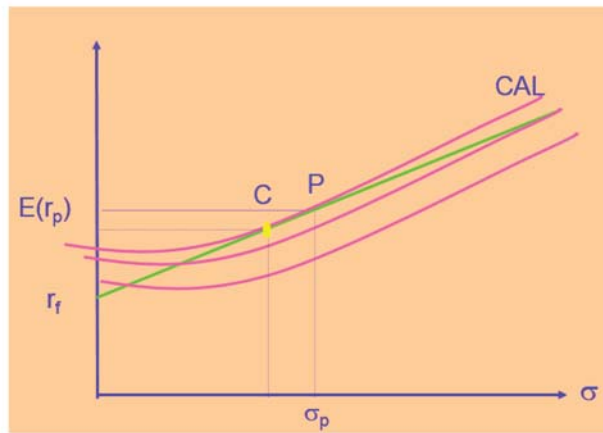
- The slope of the CAL reflects the price of risk.
- It is called the **reward-to-variability** or **reward-to-volatility** ratio.
- Usually it is called the **Sharpe ratio** (of the risky portfolio).
- It's due to William Sharpe (Nobel Prize 1990):



CAL Figure



Where Does the Optimal Combination Lie?



Analytical

- Maximize:

$$\begin{aligned} U &= \mathbb{E}(r_c) - 0.5 A \sigma_c^2 \\ &= r_f + w_p [\mathbb{E}(r_p) - r_f] - 0.5 A w_p^2 \sigma_p^2 \end{aligned}$$

- Decision variable = w_p .

Solved

$$w_p = \frac{\mathbb{E}(r_p) - r_f}{A \sigma_p^2}$$

- Proportion risky assets w_p **increases** when:
 - Risk Premium → **Increases**.
 - Risk Level → **Decreases**.
 - Risk Aversion → **Decreases**.

What is a Risk-Free Asset?

- An asset whose return is known **ex-ante**
- No investment risk
- No credit risk
- No inflation risk
- No risk whatsoever

In practice:

- Government paper
- Long-term Treasury bond
- Short-term Treasury bill (T-bill)

Why Short-Term T-Bills?

A short-term nominal interest rate equals

- Short-term real interest rate
- Expected inflation

A short-term T-bill therefore protects against expected inflation
In the short run inflation is predictable (much less in the long run)

What is "the" Risky Portfolio p ?

- It can be anything.
- It make sense to identify it to the market-capitalization weighted portfolio.
- It includes all risky assets in proportion to their value.
- It therefore reflects the total financial market.
- In practice: a stock market index that is tradable → index strategy.

Index Strategy

- Market capitalization weighted portfolio
- Index funds
- ETF: Exchange Traded Funds or 'trackers'

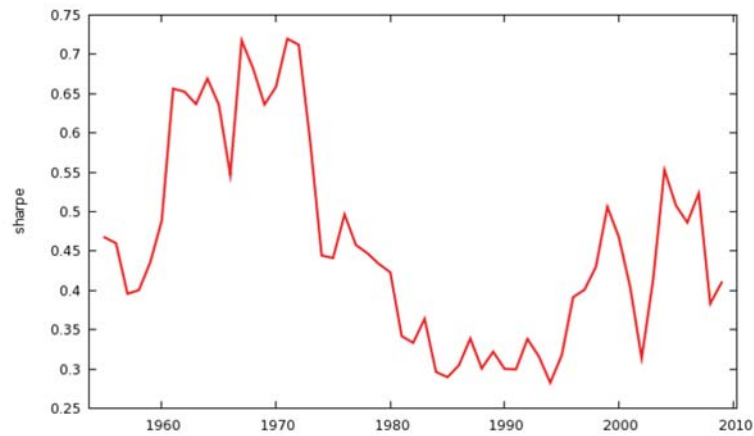
Is a **passive strategy**

- Easy
- Cheap
- A good deal?

Some Numbers

- p = US Large caps
- f = US T-bills
- Period 1926-2009
- Average excess return p : 7.93%
- Standard deviation p : 20.81%
- Sharpe ratio = 0.38
- In words: for every increase in risk by 1%, the investor gets an addition 0.38% in return.

But There is Some Time-variation (30y windows)



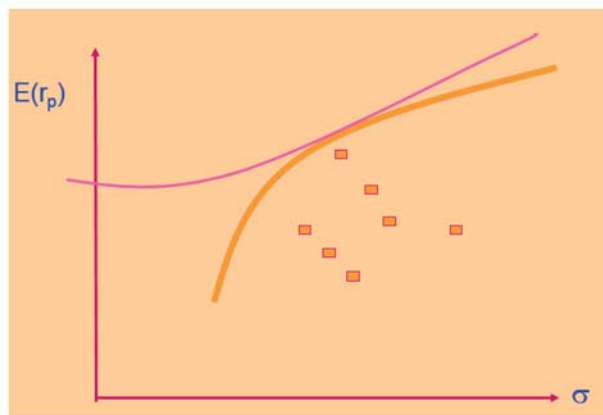
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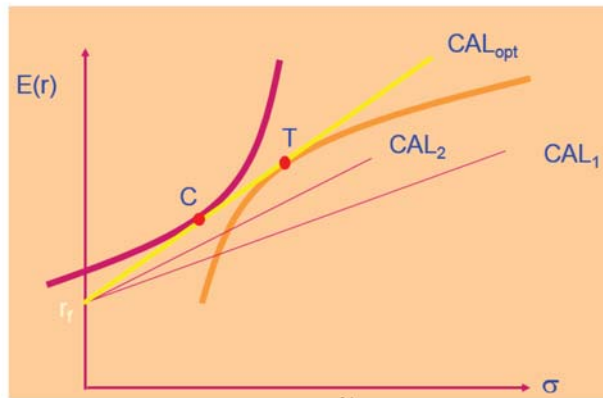
An Optimal Risky Portfolio?

- Which portfolio p do we have to consider?
- When there are N risky assets: choose the one that maximizes the investor's utility function.
- We need:
 - All expected returns (N).
 - All volatilities (N).
 - All correlations ($N(N-1)/2$).
- → the optimal risky portfolio;

The Optimal Portfolio



But What if we Add the Risk-Free Asset?



Steps to Take

- Compute:
 - expected returns.
 - standard deviations.
 - **all** correlation coefficients.
- Do NOT compute all combinations, but only the **efficient** portfolios.
- Determine the tangency portfolio.
- Determine the optimal portfolio (tangency portfolio + risk-free asset).

Tangency Portfolio

- Tangency portfolio maximizes the slope of the CAL:

$$sh_p = \frac{\mathbb{E}(r_p) - r_f}{\sigma_p}$$

- Subject to constraints.

Separation Theorem

- The optimal portfolio selection can be split up into two parts.
- A “technical” part: compute the tangency portfolio.
- Is identical for all investors (with same expectations).
- A “decision” part depends on risk aversion.
- Two portfolios are sufficient.

Factor Models

- Original purpose:
 - Simplify calculation efficient frontier
- Other (and more popular) applications:
 - valuation (APT, see later).
 - explaining expected returns.
 - consistent correlation structure.
 - risk measurement.
 - performance evaluation.

What is (or was) the problem?

Many parameters are needed to compute the entire efficient frontier

- Expected return: N
- Risk: N
- Correlation : $N(N - 1)/2$
- Total : $N(N+3)/2$

Where Can we Find the Solution?

- The large number of parameters is due to the correlation estimates
- Assume a model to explain correlations (or covariances)

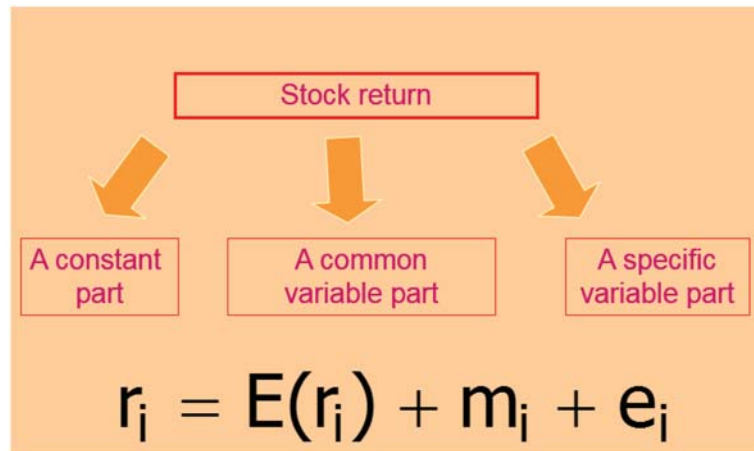
Observation:

- Correlations are usually positive
- May be due to **common factors**

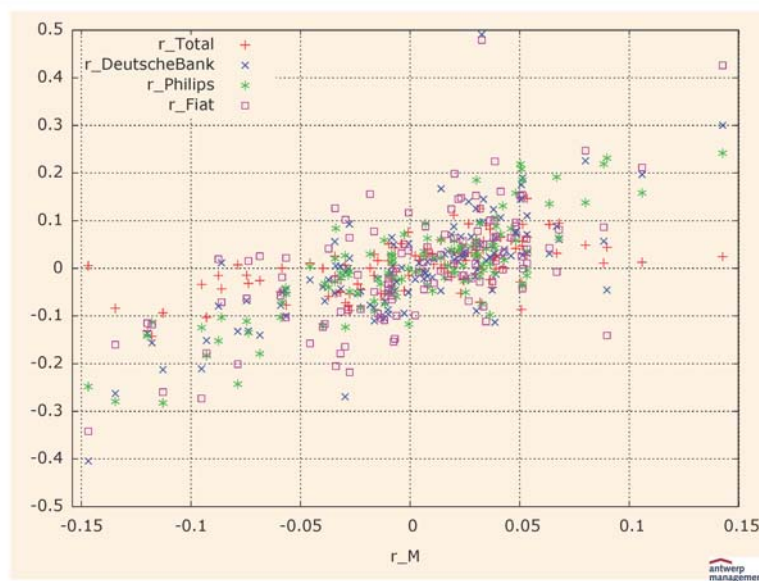
Common Factors



Principle Factor Models



Market Factor as Common Factor



Variable Part

- Return uncertainty can only be due to the variable part
- The specific part becomes negligible in a portfolio
- A factor model “explains” (or rather models) the common part
- The factor model can be used to compute portfolio risk

Implications

Return surprise is decomposed into:

- Macro surprise (m)
- Firm-specific surprise (e).

Risk can be decomposed accordingly:

$$\sigma_i^2 = \sigma_{mi}^2 + \sigma_{ei}^2$$

Covariance is only due to common risk

Single Index Model

- The common part is represented by one factor
- In the single-index model this factor is a stock market index

$$r_i - r_f = \alpha_i + \beta_i(r_M - r_f) + e_i$$

$$R_i = \alpha_i + \beta_i R_M + e_i$$

- Note the use of capital R to denote **excess returns**
- In practice, both excess returns and total returns are used in the single-index model, depending on data availability

Expected Returns

$$E(R_i) = \alpha_i + \beta_i R_M + 0$$

Risk premium decomposes into:

- Systematic risk premium $\beta_i R_M$
- Nonmarket premium α_i

Does it help for portfolio selection?

Estimates: $2N + 1$ versus N

- α_i : N
- β_i : N
- $E(R_M)$: 1

So, What About Variance?

$$\begin{aligned}\sigma_i^2 &= E[R_i - E(R_i)]^2 \\ &= E[\beta_i(R_M - E(R_M)) + e_i]^2 \\ &= \beta_i^2 \sigma_M^2 + \sigma_{ei}^2 + 0\end{aligned}$$

Total risk decomposes into:

- Systematic risk: $\beta_i^2 \sigma_M^2$
- Firm-specific risk: σ_{ei}^2

Does it help for portfolio selection?

- Estimates: $N + 1$ versus N
 - market risk: 1
 - residual risk: N

Covariances?

$$\begin{aligned}\sigma_{ij} &= E[(R_i - E(R_i))(R_j - E(R_j))] \\ &= \beta_i \beta_j \sigma_M^2 + \beta_i \text{cov}(R_M, e_i) + \beta_j \text{cov}(R_M, e_j) + \beta_i \beta_j \text{cov}(e_i, e_j) \\ &= \beta_i \beta_j \sigma_M^2\end{aligned}$$

Does it help for portfolio selection?

- Additional estimates: 0

And Correlations?

They follow from the variances and covariances:

$$\begin{aligned} \text{corr}(R_j, R_k) &= \rho_{jk} = \frac{\sigma_{jk}}{\sigma_j \sigma_k} \\ &= \frac{\beta_j \beta_k \sigma_M^2}{\sqrt{\beta_j^2 \sigma_M^2 + \sigma_{ej}^2} \sqrt{\beta_k^2 \sigma_M^2 + \sigma_{ek}^2}} \end{aligned}$$

Summary

- Number of parameters: $3N + 2$ versus $N(N + 3)/2$ for a “full” computation
- Considerable work reduction: for 1000 stocks only 3002 instead of more than 500 000 estimates. . .
- Other advantage: makes life easier for specialised analysts
- Leads to consistent correlation matrices
- Disadvantage: somewhat simplistic

The Single Index Model is a Factor Model

$$\begin{aligned} R_i &= a_i + b_i R_M + e_i \\ E(R_i) &= a_i + b_i E(R_M) + e_i \end{aligned}$$

- Subtract and rearrange

$$R_i = E(R_i) + b_i(R_M - E(R_M)) + e_i$$

- Compare with

$$R_i = E(R_i) + m_i + e_i$$

Single Index Models and Diversification

$$\begin{aligned} R_p &= w_1 R_1 + w_2 R_2 \\ &= w_1(\alpha_1 + \beta_1 R_M + e_1) + w_2(\alpha_2 + \beta_2 R_M + e_2) \\ &= (w_1 \alpha_1 + w_2 \alpha_2) + (w_1 \beta_1 + w_2 \beta_2) R_M + (w_1 e_1 + w_2 e_2) \\ &= \alpha_p + \beta_p R_M + e_p \end{aligned}$$

α , β and e are weighted averages

Portfolio Variance

$$\begin{aligned}
 \sigma_p^2 &= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_{12} \\
 &= w_1^2 (\beta_1^2 \sigma_M^2 + \sigma_{e1}^2) + w_2^2 (\beta_2^2 \sigma_M^2 + \sigma_{e2}^2) + 2w_1 w_2 \beta_1 \beta_2 \sigma_M^2 \\
 &= (w_1^2 \beta_1^2 + w_2^2 \beta_2^2 + 2w_1 w_2 \beta_1 \beta_2) \sigma_M^2 \\
 &\quad + (w_1^2 \sigma_{e1}^2 + w_2^2 \sigma_{e2}^2) \\
 &= \beta_p^2 \sigma_M^2 + (w_1^2 \sigma_{e1}^2 + w_2^2 \sigma_{e2}^2)
 \end{aligned}$$

Interpretation Last Term

Let $w_1 = w_2 = 1/2$

$$\begin{aligned}
 (w_1^2 \sigma_{e1}^2 + w_2^2 \sigma_{e2}^2) &= \frac{1}{2} \left(\frac{1}{2} \sigma_{e1}^2 + \frac{1}{2} \sigma_{e2}^2 \right) \\
 &= \frac{1}{2} \bar{\sigma}_e^2
 \end{aligned}$$

Diversification for N Assets

$$\sigma_p^2 = \beta_p^2 \sigma_M^2 + \frac{1}{N} \bar{\sigma}_e^2$$

- When N increases without bound:

$$\sigma_p^2 \approx \beta_p^2 \sigma_M^2$$

- In the limit only **beta** is important to portfolio risk

How to Obtain Estimates?

$$R = \alpha + \beta R_M + e$$

Two requirements:

- You want to explain as much as possible of return variation.
- Firm-specific “surprise” should be zero on average.

So, choose β to minimize $\text{var}(e)$:

$$\begin{aligned} \min \text{var}(e) &= \text{var}(R - \alpha - \beta R_M) \\ &= \text{var}(R) + \beta^2 \text{var}(R_M) - 2\beta \text{cov}(R, R_M) \\ \frac{d \text{var}(e)}{d\beta} &= 2\beta \text{var}(R_M) - 2\text{Cov}(R, R_M) = 0 \end{aligned}$$

$$\text{Hence } \beta = \frac{\text{cov}(R, R_M)}{\text{var}(R_M)}$$

... and Alpha

- Surprise is zero on average:

$$\begin{aligned} E(e) &= E(R - \alpha - \beta R_M) = 0 \\ \Rightarrow \alpha &= E(R) - \beta E(R_M) \end{aligned}$$

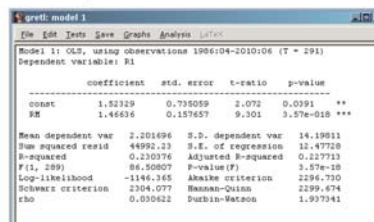
- How to estimate α and β ?
- Replace population ('true') values by sample equivalents.
 - Estimate expected values.
 - Estimate variance and covariance.
- Implication of choice:
 - Firm-specific return is unrelated to market return:

$$\begin{aligned} \text{cov}(e, R_M) &= \text{cov}(R - \alpha - \beta R_M, R_M) \\ &= \text{cov}(R, R_M) - \beta \text{var}(R_M) \\ &= \text{cov}(R, R_M) - \frac{\text{cov}(R, R_M)}{\text{var}(R_M)} \text{var}(R_M) \\ &= 0 \end{aligned}$$

In Practice

Practically, the single-index model is estimated using linear regression.

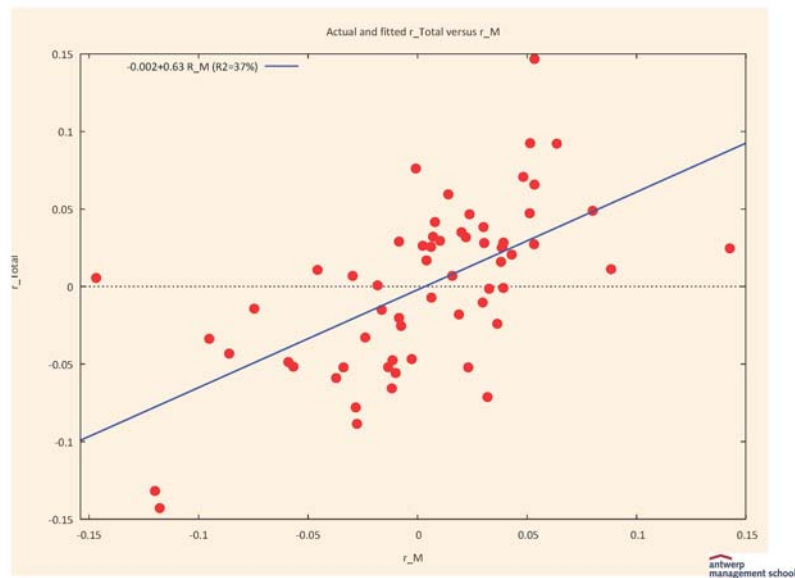
- In Gretl: use script EF Regression analysis 1:
- Choose: >Model>Ordinary Least Squares, or
- Click on "beta hat" icon, or
- In script or console `ols yvar const xvar`.



	coefficient	std. error	t-ratio	p-value
const	1.52329	0.735059	2.072	0.0391 **
R1	1.46636	0.157657	9.301	3.57e-018 ***

Mean dependent var	2.201696	S.D. dependent var	14.19811
Sum squared resid	44992.23	S.E. of regression	12.47728
R-squared	0.230376	Adjusted R-squared	0.217713
F(1, 289)	86.50807	P-value(F)	3.57e-18
Log-likelihood	-1146.365	Akaike criterion	2296.730
Schwarz criterion	2304.077	Hannan-Quinn	2299.674
zho	0.030622	Durbin-Watson	1.937041

Example Security Characteristic Line



The Numbers for Our Example

Asset	Beta	Rsq
Total	0.62	35%
Deutsche	1.59	52%
Philips	1.60	65%
Fiat	1.52	39%
Port 1	1.11	64%
Port 2	1.56	69%
All	1.33	78%

Stable Betas?

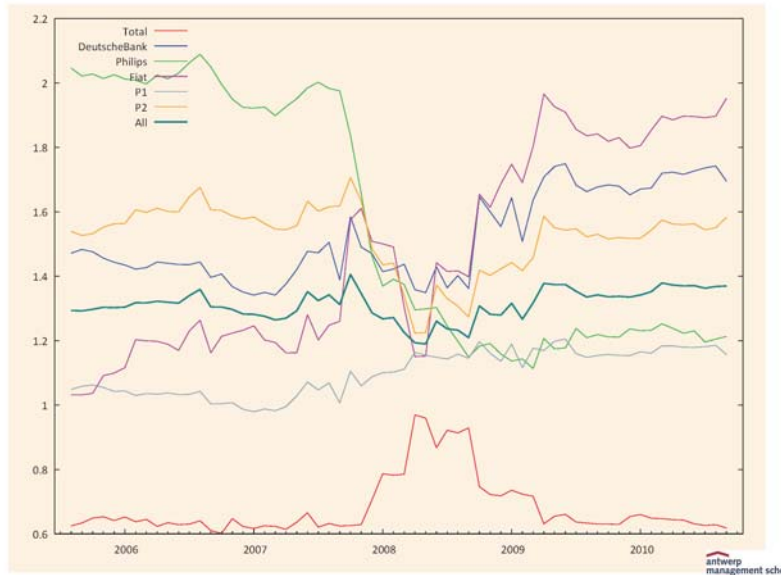
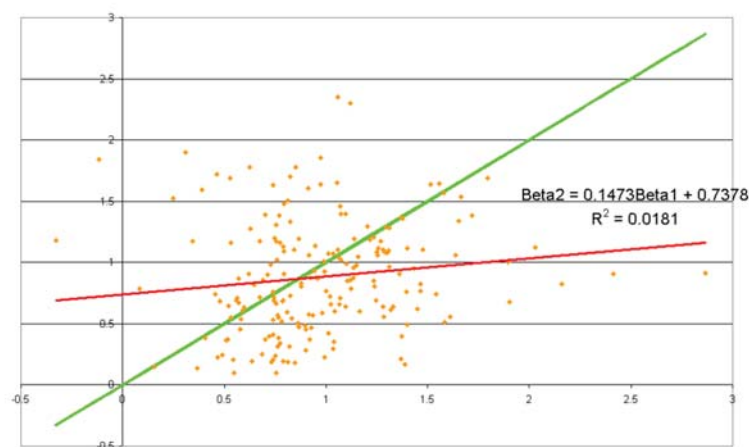


Illustration: European Stocks



Outline

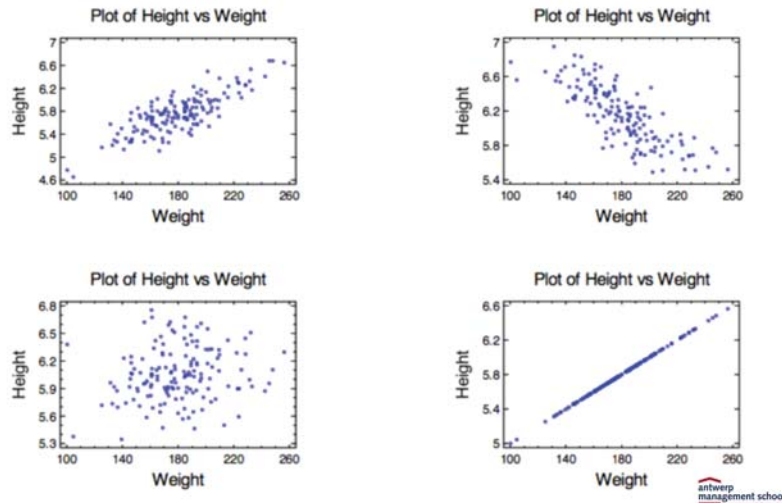
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Learning Objectives

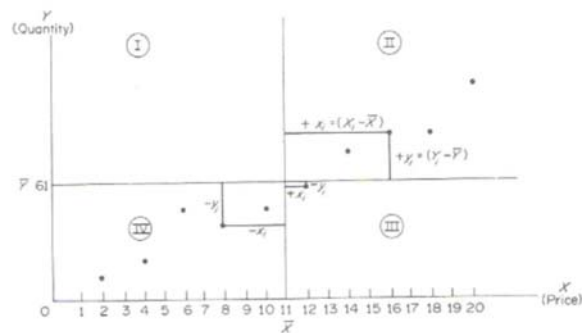
- Understand covariance and correlation and their limitations
- Understand the simple linear regression model
- Assess how good the regression is
- Be able to test the significance of the coefficients
- Perform regression in Excel

Scatter Plots

A scatter plot (or scatter diagram) can be used to show the relationship between two variables



Measuring Comovement: Covariance



$$\text{cov}(X, Y) = \sigma_{XY} \equiv E(X - \mu_X)(Y - \mu_Y)$$

$$\text{cov}(X, Y) = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{n-1}$$

Sample Covariance:

Measuring Comovement: Covariance

Plotting a scatter diagram between two variables, we may observe

- a positive relationship,
- a negative relationship or
- no relationship at all.

If we can discern a relationship, the relationship can be

- linear or
- nonlinear

Covariance measures the the **degree of co-movement** between two variables

Covariance: Calculation Rules

1. Let $Y = V + W$ then $\text{cov}(X, V + W) = \text{cov}(X, V) + \text{cov}(X, W)$

$$\begin{aligned}
 \text{cov}(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\
 &= E[(X - \mu_X)(V + W - \mu_V - \mu_W)] \\
 &= E[(X - \mu_X)(V - \mu_V) + (X - \mu_X)(W - \mu_W)] \\
 &= E[(X - \mu_X)(V - \mu_V)] + E[(X - \mu_X)(W - \mu_W)] \\
 &= \text{cov}(X, V) + \text{cov}(X, W)
 \end{aligned}$$

Obviously, if $Y = U + V + W$ then

$$\text{cov}(X, Y) = \text{cov}(X, U) + \text{cov}(X, V) + \text{cov}(X, W).$$

Covariance: Calculation Rules

2. If $Y = b$ then $\text{cov}(X, b) = 0$

$$\begin{aligned}\text{cov}(X, Y) &= \text{cov}(X, b) \\ &= E[(X - \mu_X)(b - b)] \\ &= 0\end{aligned}$$

Covariance: Calculation Rules

3. If $Y = bZ$ then $\text{cov}(X, bZ) = b \text{cov}(X, Z)$

$$\begin{aligned}\text{cov}(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= E[(X - \mu_X)(bZ - b\mu_Z)] \\ &= b E[(X - \mu_X)(Z - \mu_Z)] \\ &= b \text{cov}(X, Z)\end{aligned}$$

Measuring Comovement: Correlation

- Covariance breaks down as a good measure of co-movement since is **not scale-invariant**.
- Simply measure stock returns in basis points instead of in percentage points and you will see the covariance changing dramatically.
- In order to create a measure that solves this deficiency, we can simply divide the covariance by the standard deviations of the two random variables which gives us the **correlation coefficient**:

$$\rho_{X,Y} = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}.$$

Measuring Comovement: Correlation

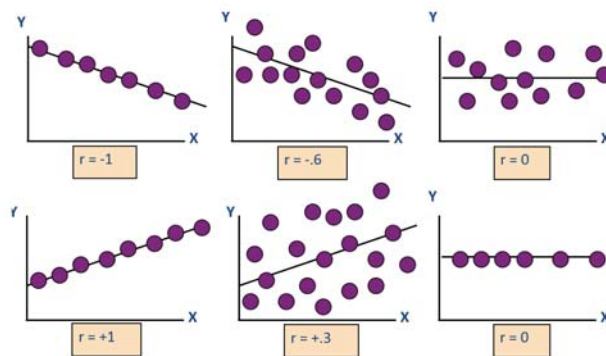
- The correlation coefficient can be shown to take values in the interval $[-1, +1]$.
- The correlation is a measure of linear correlation which is invariant to linear transformations.

$$\begin{aligned} \rho_{a+bX, c+dY} &= \frac{\text{cov}(a + bX, c + dY)}{\sqrt{\text{var}(a + bX)} \sqrt{\text{var}(c + dY)}} \\ &= \frac{bd \text{cov}(X, Y)}{\sqrt{b^2 \text{var}(X)} \sqrt{d^2 \text{var}(Y)}} \\ &= \frac{bd \text{cov}(X, Y)}{bd \sqrt{\text{var}(X)} \sqrt{\text{var}(Y)}} \\ &= \rho_{X,Y} \end{aligned}$$

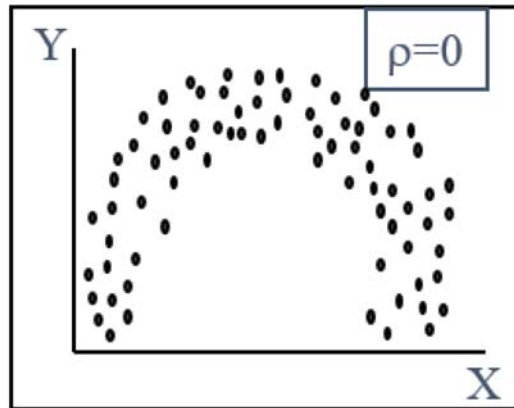
Interpretation

- The closer to -1 , the stronger the negative linear relationship
- The closer to 1 , the stronger the positive linear relationship
- The closer to 0 , the weaker any positive linear relationship

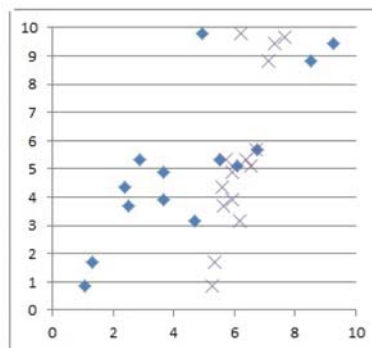
Degrees of linear correlation



Degrees of linear correlation

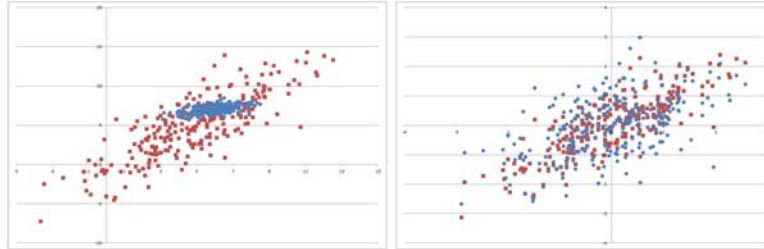


Do Not Eyeball



Note: The “blue diamonds” and the “crosses” exhibit the same correlation

Do Not Eyeball



Note: The raw data (left) suggest the “blue dots” to be more correlated than the “red dots”. The standardized data (right) reveal the (true) opposite.

Correlation versus Regression

Correlation

- Correlation analysis is used to measure strength of the association (linear relationship) between two variables
- Correlation is only concerned with strength of the relationship
- NO CAUSAL EFFECT is implied with correlation

Regression analysis is used to

- Predict the value of a **dependent variable** based on the value of at least one **independent variable**
- Explain the impact of changes in an **independent variable** on the **dependent variable**

Causality

In general, making an exception for dynamic models, **regression analysis can not prove causality**. It is economic theory that suggest a certain relationship between variables.

This point has also be made very clearly by Kendall and Stuart (1961, vol2, Chap. 26, p 279):

"A statistical relationship, however strong and however suggestive, can never establish causal connection: our ideas of causation must come from outside statistics, ultimately from some theory or other".

What is Econometrics?

- Ragnar Frisch (Econometrica, 1933)

"its main object shall be to promote studies that aim at a unification of the theoretical-quantitative and the empirical-quantitative approach to economic problems and that are penetrated by constructive and rigorous thinking similar to that which has come to dominate the natural sciences. But there are several aspects of the quantitative approach to economics, and no single one of these aspects taken by itself, should be confounded with econometrics. Thus econometrics is by no means the same as economic statistics. Nor is it identical with what we call general economic theory, although a considerable portion of this theory has a definitely quantitative character. Nor should econometrics be taken as synonymous with the application of mathematics to economics. Experience has shown that each of these three viewpoints, that of statistics, economic theory and mathematics is necessary, but not by itself sufficient, condition for a real understanding of the quantitative relations in modern economic life. It is the unification of all three that is powerful. And it is the unification that constitutes econometrics."

Modern Definitions of Econometrics

- “the study of the of the methods that enable us to quantify economic relationships using actual data”.
- “the **science and art** of using economic theory and statistical techniques to analyze economic data”.

Data

- *Cross-sectional data* measure a certain characteristic of a sample of individuals, households, firms, or any other type of units at a specific point in time. Minor timing differences are often ignored. The order in which the data are stored in the database does not matter.
- *Time series data* consist of observations of a variable over time. The frequency can be annual, quarterly, monthly, weekly, daily. Annual data are said to have a low frequency, daily data have a high frequency. The order of the data has to be chronological. If one would scramble the data, trends and seasonal patterns would be destroyed.
- *Panel data* (also known as *longitudinal data*) consist out of a time series for each cross-sectional member in the set.
- When the data consist out of a different cross-sectional sample for each moment in time, we talk about *pooled data*.

Approaches

- Start from an existing economic theory that conjectures certain relationships between economic variables. The purpose of econometrics is to **test the validity of the theory**.
- By studying the statistical characteristics of the data, stylized facts are being found. These stylized facts become **constraints for the development of economic theory**.

Simple and Multiple Linear Regression

Simple or Univariate Regression

- Only one independent variable, X
- Relationship between X and Y is described by a linear function
- Changes in Y are assumed to be caused by changes in X

Multiple Regression

- More than one independent variable, X_i
- Relationship between X_i 's and Y is described by a linear function
- Changes in Y are assumed to be caused by changes in X_i 's

Key Assumption: Linearity in the Parameters

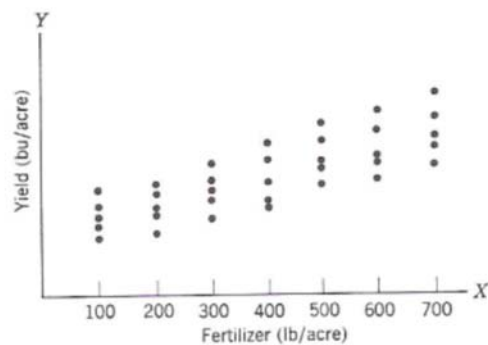
Linearity (in the parameters)

- $Y = \alpha + \beta X$
- $Y = \alpha + \beta X^2$
- $Y = \alpha + \beta \ln(X)$

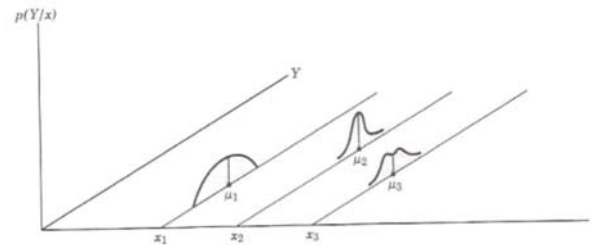
Non-linear in the parameters

- $Y = (\alpha + X)^2$
- $Y = \alpha + \beta \gamma X^\gamma$

Regression is Modelling Conditional Expectation



Regression is Modelling Conditional Expectation



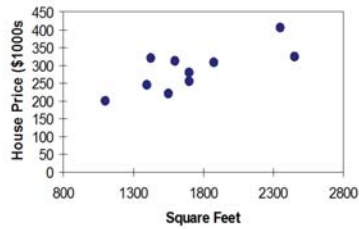
Example

- A real estate agent wishes to examine the relationship between the selling price of a home and its size (measured in square feet)
- A random sample of 10 houses is selected
- Dependent variable (Y) = house price in \$1000s Independent variable (X) = square feet

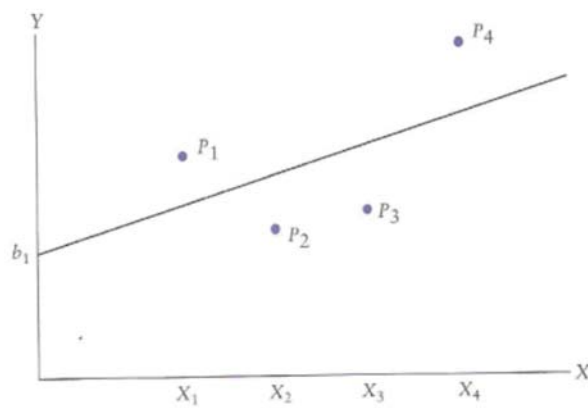
House Price in \$1000s (Y)	Square Feet (X)
245	1400
312	1600
279	1700
308	1875
199	1100
219	1550
405	2350
324	2450
319	1425
255	1700

Example: Scatter Plot

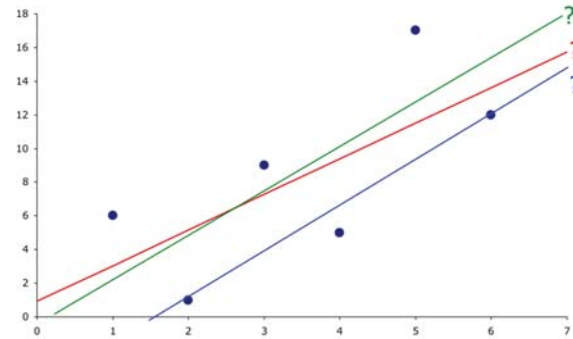
House price model: scatter plot



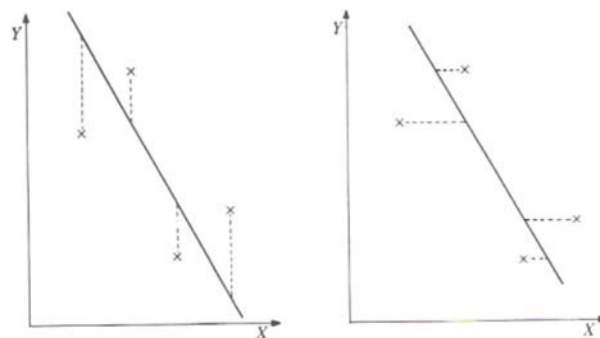
Fitted Regression Line



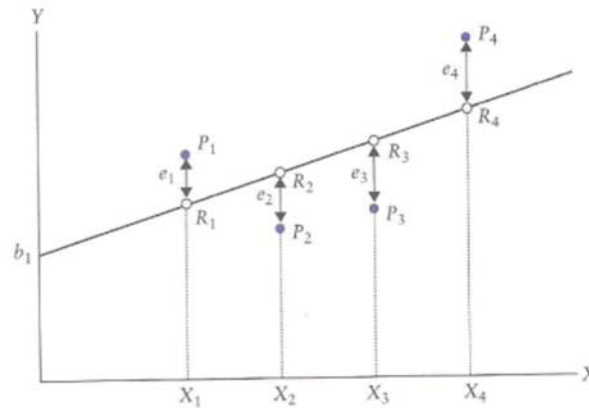
Which Line has the Best Fit?



Measuring Closeness



Fitted Regression Line



The Regression Equations

- Population

- $Y_i = a + bX_i + e_i$
- $E(Y_i|X_i) = a + bX_i$

- Sample

- $Y_i = \alpha + \beta X_i + \epsilon_i$
- $E(Y_i|X_i) = \alpha + \beta X_i$ or simply $\hat{Y}_i = \alpha + \beta X_i$
- \hat{Y}_i is the predicted value of Y_i for X_i

The Regression Equations

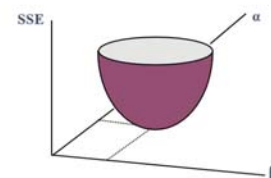
- The residuals are $\epsilon_i = Y_i - \alpha - \beta X_i$
- Measuring closeness involves some function of the residuals:
 - $\min \sum \epsilon_i$???
 - $\min \sum |\epsilon_i| \Rightarrow$ Least Absolute Deviation
 - $\min \sum \epsilon_i^2 \Rightarrow$ minimizing the sum of squared errors (SSE) i.e. (Ordinary) Least Squares

The Regression Equations

- $\min_{\alpha, \beta} SSE = \min_{\alpha, \beta} \sum \epsilon_i^2 = \min_{\alpha, \beta} \sum (Y_i - \alpha - \beta X_i)^2$
- Least Squares **Estimators**:

$$\beta = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2}$$

$$\alpha = \bar{Y} - \beta \bar{X}$$



Note: An estimator is a formula to estimate an unknown population parameter from a sample.

Example: Regression Output

Regression Statistics

Multiple R	0.76211
R Square	0.58082
Adjusted R Square	0.52842
Standard Error	41.33032
Observations	10

The regression equation is:

house price = 98.24833 + 0.10977 (square feet)

ANOVA

	df	SS	MS	F	Significance F
Regression	1	18934.9348	18934.9348	11.0848	0.01039
Residual	8	13665.5652	1708.1957		
Total	9	32600.5000			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
Intercept	98.24833	58.03348	1.69296	0.12892	-35.57720	232.07386
Square Feet	0.10977	0.03297	3.32938	0.01039	0.03374	0.18580

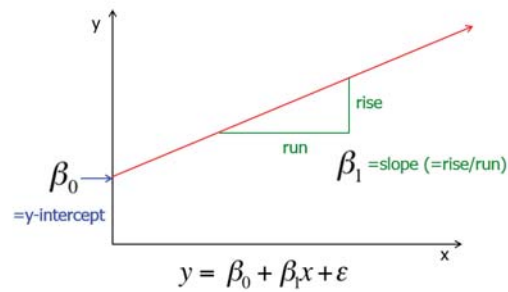


Interpretation of Slope and Intercept

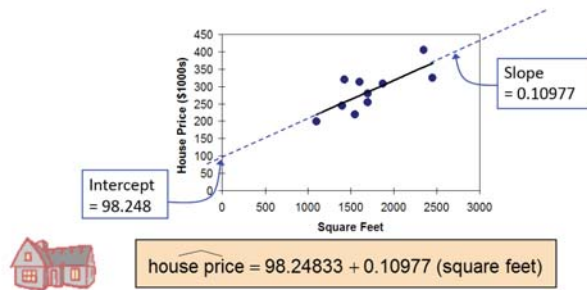
Meaning of β_0 and β_1

$\beta_1 > 0$ [positive slope]

$\beta_1 < 0$ [negative slope]



Interpretation of Slope and Intercept



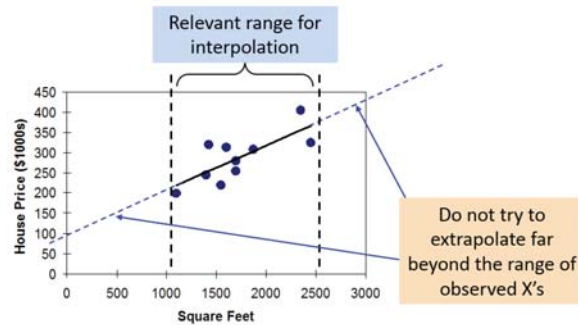
Prediction (Intrapolation)

Predict the price for a house with 2000 square feet:

$$\begin{aligned}\widehat{\text{house price}} &= 98.25 + 0.1098 (\text{sq.ft.}) \\ &= 98.25 + 0.1098(2000) \\ &= 317.85\end{aligned}$$

The predicted price for a house with 2000 square feet is 317.85(\$1,000s) = \$317,850

Prediction (Extrapolation)



How Good is the Fit?

- The least squares method will always produce a straight line, even if there is no relationship between the variables, or if the relationship is something other than linear.
- Hence, in addition to determining the coefficients of the least squares line, we need to assess it to see how well it “fits” the data.
- The evaluation methods are based on the sum of squares for errors (SSE).

How Good is the Fit?

The standard deviation of the variation of observations around the regression line is estimated by

$$S_{YX} = \sqrt{\frac{SSE}{n-2}} = \sqrt{\frac{\sum_{i=1}^n (Y_i - \hat{Y}_i)^2}{n-2}}$$

Where

SSE = error sum of squares

n = sample size

How Good is the Fit?

Regression Statistics

Multiple R	0.76211
R Square	0.58082
Adjusted R Square	0.52842
Standard Error	41.33032
Observations	10

$$S_{YX} = 41.33032$$

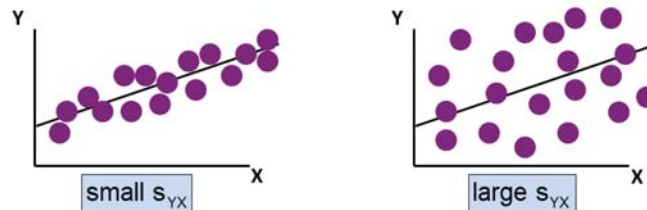
ANOVA					
	df	SS	MS	F	Significance F
Regression	1	18934.9348	18934.9348	11.0848	0.01039
Residual	8	13665.5652	1708.1957		
Total	9	32600.5000			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
Intercept	98.24833	58.03348	1.69296	0.12892	-35.57720	232.07386
Square Feet	0.10977	0.03297	3.32938	0.01039	0.03374	0.18580



How Good is the Fit?

S_{YX} is a measure of the variation of observed Y values from the regression line

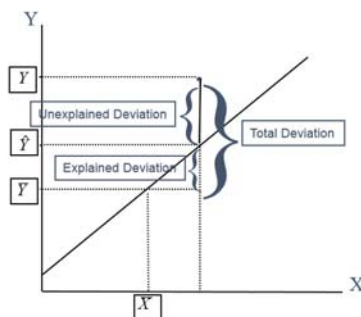


The magnitude of S_{YX} should always be judged relative to the size of the Y values in the sample data

i.e., $S_{YX} = \$41.33K$ is moderately small relative to house prices in the \$200 - \$300K range

How Good is the Fit?

The **coefficient of determination**, r^2 , is a descriptive measure of the strength of the regression relationship, a measure of how well the regression line fits the data.



$$(y - \bar{y}) = (y - \hat{y}) + (\hat{y} - \bar{y})$$

Total	=	Unexplained	Explained
Deviation		Deviation	Deviation
		(Error)	(Regression)

$$\sum (y - \bar{y})^2 = \sum (y - \hat{y})^2 + \sum (\hat{y} - \bar{y})^2$$

$$SST = SSE + SSR$$

$$r^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

Percentage of total variation explained by the regression.

The coefficient of determination is the portion of the total variation in the dependent variable that is explained by variation in the independent variable

Note: $0 \leq R^2 \leq 1$

How Good is the Fit?

Regression Statistics	
Multiple R	0.76211
R Square	0.58082
Adjusted R Square	0.52842
Standard Error	41.33032
Observations	10

$$r^2 = \frac{SSR}{SST} = \frac{18934.9348}{32600.5000} = 0.58082$$

58.08% of the variation in house prices is explained by variation in square feet

ANOVA		df	SS	MS	F	Significance F
Regression		1	18934.9348	18934.9348	11.0848	0.01039
Residual		8	13665.5652	1708.1957		
Total		9	32600.5000			

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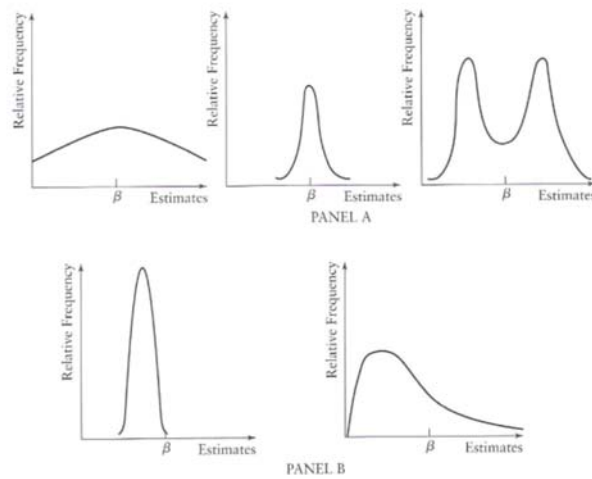


Desired Properties of an Estimator (I)

Unbiasedness:

- Suppose you evaluate an estimator (say β) many times over repeated randomly drawn samples
- For each sample you would find a different estimate (i.e. β) of the true population parameter b
- You would at least hope that the average of your estimates would give you the right answer and that you would recover the population parameter b .
- Mathematically the estimator is unbiased if $E(\beta) = b$.
- The difference, if any, between $E(\beta)$ and b is called the bias of the estimator.

Unbiased (Panel A) and Biased (Panel B) Estimators



Assumptions Needed for the OLS estimator to be Unbiased

- Linearity in the parameters
- X is not random and must take at least two different values
- Zero conditional mean (i.e. $E(e_i|X_i) = 0$) \implies adding a constant to the regression will ensure this condition is satisfied

There can be many Unbiased Estimators

The OLS estimator of the slope is not the only unbiased estimator of the slope.

If we simply estimate the slope as $\check{\beta} = \frac{Y_N - Y_1}{X_N - X_1}$, we are also using an unbiased estimator.

We know that $Y_1 = \alpha + \beta X_1 + u_1$ and $Y_N = \alpha + \beta X_N + u_N$. Substituting these expressions in the estimator we obtain,

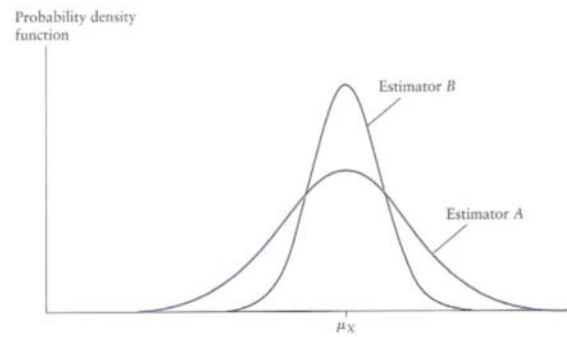
$$\begin{aligned}\check{\beta}_2 &= \frac{\alpha + \beta X_N + u_N - \alpha - \beta X_1 - u_1}{X_N - X_1} \\ &= \frac{\beta(X_N - X_1) + (u_N - u_1)}{X_N - X_1} \\ &= \beta + \frac{(u_N - u_1)}{X_N - X_1} \\ &= \beta\end{aligned}$$

Desired Properties of an Estimator (II)

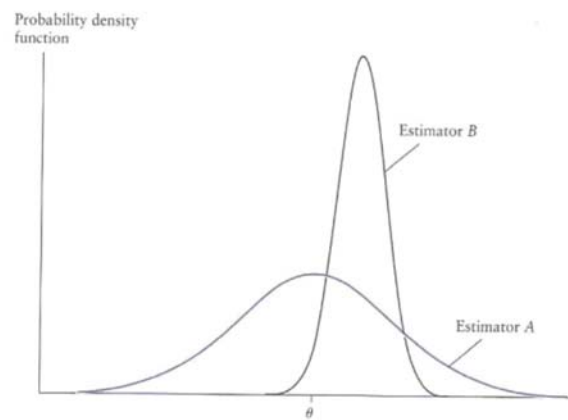
Efficiency:

- Suppose you have two estimators $\hat{\beta}$ and $\check{\beta}$ that are both unbiased. In that case you would prefer **the estimator with the smallest variance**.
- This estimator is said to use the information in the data more efficiently than the other.
- From this definition it is clear that efficiency is a relative concept vis-à-vis another (class of) estimators.

Efficient and Less Efficient Estimators



Efficient and Less Efficient Estimators



Choosing between Estimators

- An absolute preference for either unbiasedness or efficiency
- An (arbitrary) loss function that attaches weights to the bias and to the variance of the estimator
 - Example: mean squared error
 - Suppose we are estimating a population parameter b with an estimator β which has an expected value of $\bar{\beta}$
 - The bias can be expressed as $(b - \bar{\beta})$
 - the mean squared error (MSE) can be computed as follows:

$$\begin{aligned}
 MSE &= E(\beta - b)^2 \\
 &= E(\beta - \bar{\beta} + \bar{\beta} - b)^2 \\
 &= E[(\beta - \bar{\beta}) + (\bar{\beta} - b)]^2 \\
 &= E(\beta - \bar{\beta})^2 + E(\bar{\beta} - b)^2 + 2E(\beta - \bar{\beta})(\bar{\beta} - b) \\
 &= E(\beta - \bar{\beta})^2 + E(\bar{\beta} - b)^2 \\
 &= \text{variance of the estimator} + \text{bias}^2
 \end{aligned}$$

Assumptions Needed to Calculate the Standard Errors of the Estimates

- SR1 Linearity in the parameters
- SR2 X is not random and must take at least two different values
- SR3 Zero conditional mean (i.e. $E(e_i|X_i) = 0$) \implies adding a constant to the regression will ensure this condition is satisfied
- SR4 Homoskedasticity (i.e. $\text{var}(e) = \sigma^2 = \text{var}(Y)$)
- SR5 No Autocorrelation (i.e. $\text{cov}(e_i, e_j) = \text{cov}(Y_i, Y_j) = 0$)

The Precision of the Estimators

If the assumptions stated hold the variances of the OLS estimators are

$$\text{var}(\alpha) = \sigma^2 \left[\frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \right]$$

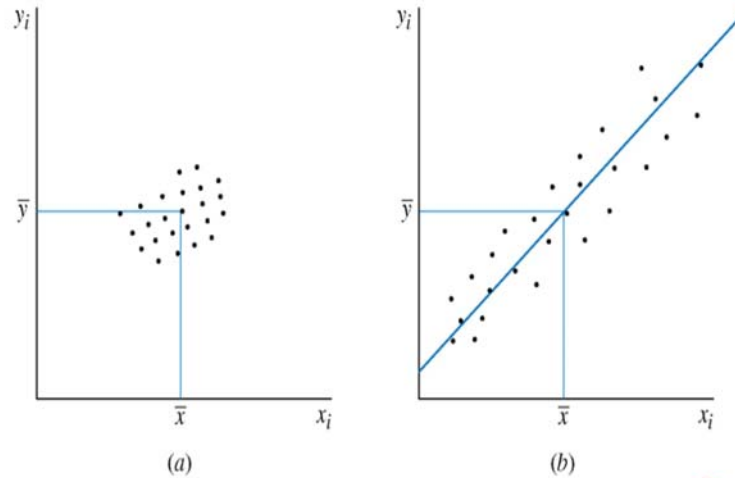
$$\text{var}(\beta) = \frac{\sigma^2}{\sum (X_i - \bar{X})^2}$$

A Precise Estimator when ...

- The **smaller the variance term** σ^2 , the smaller the uncertainty there is in the statistical model, and the smaller the variances of the least squares estimators
- The **larger the sum of squares**, $\sum (X_i - \bar{X})^2$, the smaller the variances of the least squares estimators and the more precisely we can estimate the unknown parameters
- The **larger the sample size**, N , the smaller the variances and covariance of the least squares estimators
- The **smaller the term** $\sum X_i^2$ the smaller the variance of the least squares estimator α

Illustrating Precision

(a) Low X variation, low precision (b) High X variation, high precision



The Precision of the Estimates

Regression Statistics

Multiple R	0.76211
R Square	0.58082
Adjusted R Square	0.52842
Standard Error	41.33032
Observations	10

ANOVA

	df	SS	MS	F	Significance F
Regression	1	18934.9348	18934.9348	11.0848	0.01039
Residual	8	13665.5652	1708.1957		
Total	9	32600.5000			

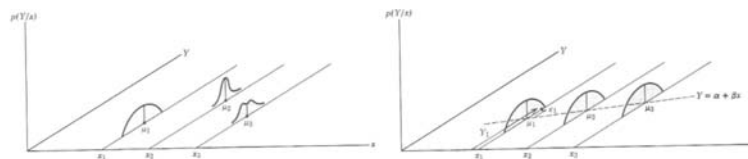
	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
Intercept	98.24833	58.03348	1.69296	0.12892	-35.57720	232.07386
Square Feet	0.10977	0.03297	3.32938	0.01039	0.03374	0.18580



The Gauss-Markov Theorem

- Under the assumptions SR1-SR5 of the simple linear regression model, the estimators α and β have the smallest variance of all linear and unbiased estimators of a and b .
- They are called the **Best Linear Unbiased Estimators (BLUE)** of a and b

Regression is Modelling Conditional Expectation



The Distribution of the OLS Estimators

If we make the assumption that the errors are normally distributed then the least squares estimators are normally distributed for **any sample**:

$$\alpha \sim t \left(a, \frac{\sigma^2 \sum X_i^2}{N \sum (X_i - \bar{X})^2}, n-2 \right)$$

$$\beta \sim t \left(b, \frac{\sigma^2}{\sum (X_i - \bar{X})^2}, n-2 \right)$$

This allows us to perform statistical inference about the significance of the parameters

The rejection region depends on whether or not we're doing a one- or two-tail test (two-tail test is most typical)

Inference

$$H_0: \beta = 0$$

$$H_1: \beta \neq 0$$

From Excel output:

	Coefficients	Standard Error	t Stat	P-value
Intercept	98.24833	58.03348	1.69296	0.12892
Square Feet	0.10977	0.03297	3.32938	0.01039

$$t = \frac{\beta - b}{S_\beta} = \frac{0.10977 - 0}{0.03297} = 3.32938$$

$$df = n - 2 = 8 \quad 2 \text{ tails}$$

$$\text{TDIST}(3.32938, 8, 2) = .010393985$$

Inference

$$H_0: \beta = 0 \quad H_A: \beta \neq 0$$

Test statistic

$$t = \frac{\beta - b}{S_\beta}$$

where:

β = regression slope coefficient

b = hypothesized slope

S_β = standard error of the slope

$$p\text{-value} = \text{TDIST}(t_{n-2,2})$$

$$\text{d.f.} = n - 2$$

Inference

Critical Values						
Normal	0.05%	-3.291				
	1%	-2.326				
	2.50%	-1.960				
	5%	-1.645				
	10%	-1.282				
One Tailed = t.inv						
Student	0.05%	5	8	20	30	100
	1%	-6.869	-5.041	-3.850	-3.646	-3.390
	1%	-3.365	-2.896	-2.528	-2.457	-2.364
	2.50%	-2.571	-2.306	-2.086	-2.042	-1.984
	5%	-2.015	-1.860	-1.725	-1.697	-1.660
	10%	-1.476	-1.397	-1.325	-1.310	-1.290
Two Tailed = t.inv.2T						
Student	0.05%	5	8	20	30	100
	0.05%	7.976	5.617	4.146	3.902	3.598
	1%	4.032	3.355	2.845	2.750	2.626
	2.50%	3.163	2.752	2.423	2.360	2.276
	5%	2.571	2.306	2.086	2.042	1.984
	10%	2.015	1.860	1.725	1.697	1.660

Inference

(continued)

Test Statistic: $t = 3.329$

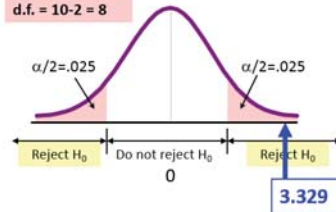
$$H_0: b = 0$$

$$H_1: b \neq 0$$

From Excel output:

	Coefficients	Standard Error	t Stat	P-value
Intercept	98.24833	58.03348	1.69296	0.12892
Square Feet	0.10977	0.03297	3.32938	0.01039

d.f. = 10 - 2 = 8



Decision:

Reject H_0 at $\alpha = .05$

Conclusion:

There is sufficient evidence
that square footage affects
house price

Inference

Confidence Interval Estimate of the Slope:

$$b = \beta \pm t_{n-2} S_{\beta}$$

d.f. = n - 2

Excel Printout for House Prices:

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
Intercept	98.24833	58.03348	1.69296	0.12892	-35.57720	232.07386
Square Feet	0.10977	0.03297	3.32938	0.01039	0.03374	0.18580

At 95% level of confidence, the confidence interval for the slope b is (0.0337, 0.1858)

Inference

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
Intercept	98.24833	58.03348	1.69296	0.12892	-35.57720	232.07386
Square Feet	0.10977	0.03297	3.32938	0.01039	0.03374	0.18580

Since the units of the house price variable is \$1000s, we are 95% confident that the average impact on sales price is between \$33.70 and \$185.80 per square foot of house size

This 95% confidence interval **does not include 0**.

Conclusion: There is a significant relationship between house price and square feet at the .05 level of significance

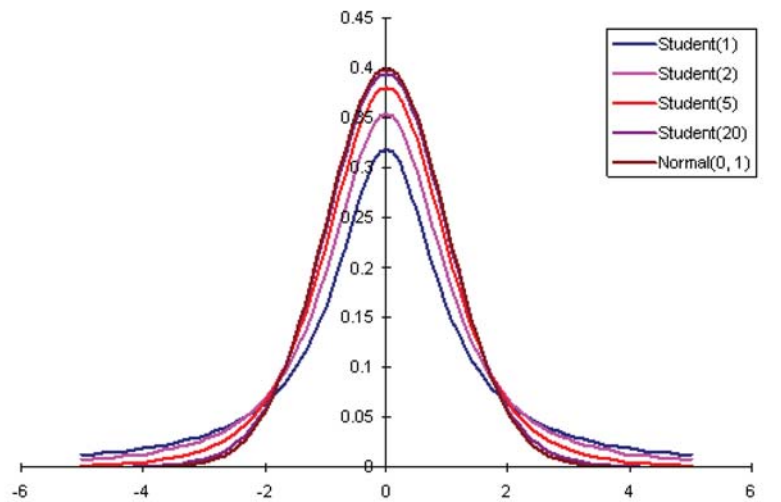
The Distribution of the OLS Estimators

If we make the assumption that the errors are normally distributed then the least squares estimators are normally distributed for **large samples**:

$$\alpha \sim N\left(a, \frac{\sigma^2 \sum X_i^2}{N \sum (X_i - \bar{X})^2}\right)$$

$$\beta \sim N\left(b, \frac{\sigma^2}{\sum (X_i - \bar{X})^2}\right)$$

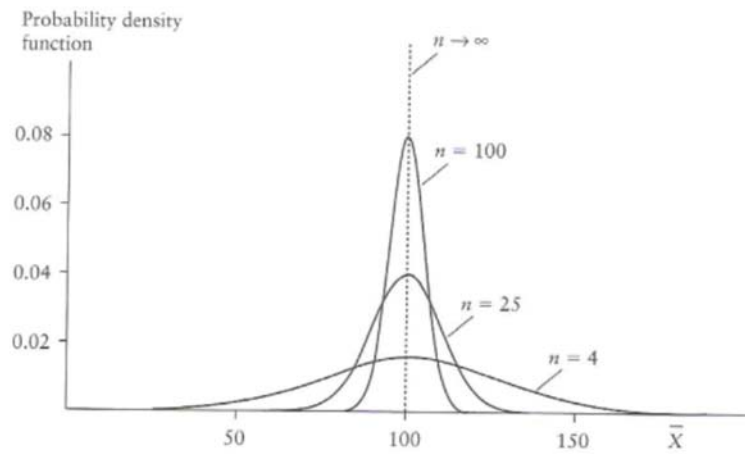
The Distribution of the OLS Estimators



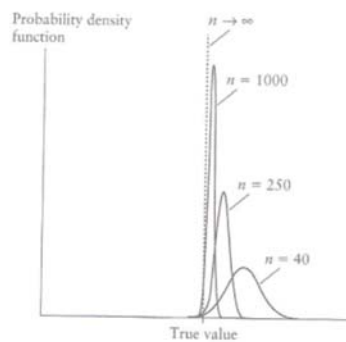
Desired Properties of an Estimator (III)

Consistency: Suppose you increase the sample size, what would you expect that happens to the distribution of your estimator?
You would expect that the probability that the estimator will yield estimates within a very small interval of the true value of the population parameter will approach

Illustrating Consistency



Note that a Biased Estimator can be Consistent.



Regression Diagnostics

There are three conditions that are required in order to perform a simple regression analysis.

- The error variable must be normally distributed,
- The error variable must have a constant variance,
- The errors must be independent of each other.

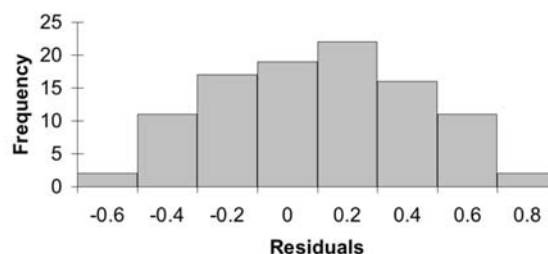
How can we diagnose violations of these conditions?

Residual Analysis (i.e. examine the differences between the actual data points and those predicted by the linear equation)

If one of these conditions is violated, OLS estimators are not BLUE and adjusted estimation procedures are needed.

Non-normality

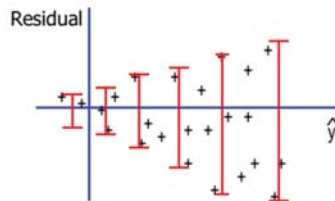
We can take the residuals and put them into a histogram to visually check for normality...



...we're looking for a bell shaped histogram with the mean close to zero [our old "test for normality"]. ✓

Homoskedasticity

When the **requirement of a constant variance** is violated, we have a condition of **heteroscedasticity**.



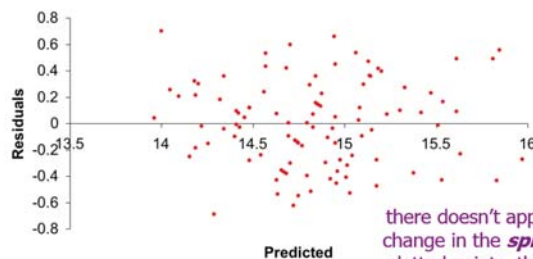
The spread increases with \hat{y}

We can diagnose heteroscedasticity by plotting the residual against the predicted y .

Homoskedasticity

If the variance of the error variable (σ_e^2) is not constant, then we have “**heteroscedasticity**”. Here’s the plot of the residual against the predicted value of y :

Plot of Residuals vs Predicted



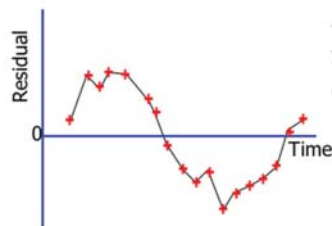
there doesn't appear to be a change in the **spread** of the plotted points, therefore **no heteroscedasticity** ✓

Absence of Autocorrelation

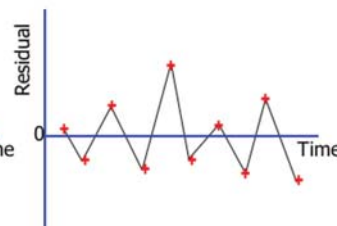
- If we were to observe the auction price of cars every week for, say, a year, that would constitute a time series.
- When the data are time series, the errors often are correlated. Error terms that are correlated over time are said to be autocorrelated or serially correlated.
- We can often detect autocorrelation by graphing the residuals against the time periods. If a pattern emerges, it is likely that the independence requirement is violated.

Absence of Autocorrelation

Patterns in the appearance of the residuals over time indicates that autocorrelation exists:



Note the runs of positive residuals, replaced by runs of negative residuals



Note the oscillating behavior of the residuals around zero.

Outline

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Equilibrium Pricing Models

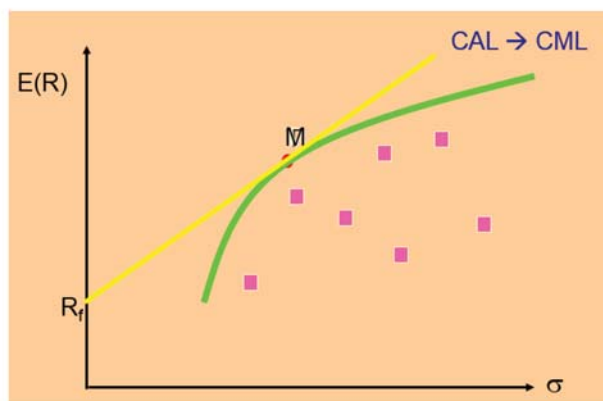
- Pricing model
- Relation risk - expected return
- CAPM and APT
- Applications
 - ex-ante investment evaluation
 - pricing new listings
 - ex-post investment evaluation
 - cost of capital investments

Capital Asset Pricing Model

Hypotheses:

- “Perfect” Market
- “Homogeneous” investors
 - use Markowitz
 - different wealth
 - different risk-aversion

Step 1: The Market Portfolio



Therefore...

- The **market portfolio** is MV-efficient: index strategy ('mutual fund theorem')
- Everybody invests partly in M
- Differences are due to different risk aversion
- The CAL is the **Capital Market Line**
- Pricing is relative with respect to M

The Capital Market Line

- The CML depicts the relation between risk and return for **efficient** portfolios
- The **price for time** is given by the intercept in the CML: r_f
- The **price for risk** is given by the CML slope:

$$\frac{E(r_M) - r_f}{\sigma_M}$$

Step 2: The Risk Premium for M

- Recall the optimal proportion of risky assets:

$$w_p = \frac{E(r_p) - r_f}{A \sigma_p^2}$$

- In equilibrium is $w_M = 100\%$ so

$$E(r_M) - r_f = \bar{A} \sigma_M^2$$

Therefore ...

The risk premium for the market is...

- proportional to the **average risk aversion** \Rightarrow INVESTORS
- proportional to the **total risk** of the market portfolio \Rightarrow ECONOMY

Step 3: Risk Premium for Individual Assets

- Risk = asset's contribution in M
- Diversified portfolio: contribution is only systematic risk
- Here: systematic risk = β
- Risk premium is proportional to β

$$E(r_i - r_f) = \beta E(r_M - r_f)$$

Therefore ...

$$\beta_i = \frac{\text{cov}(r_i, r_M)}{\sigma_M^2}$$

Definition of beta

$$\beta_P = \sum w_i \beta_i$$

Beta of portfolio is weighted average

$$\beta_M = 1$$

Average beta is 1

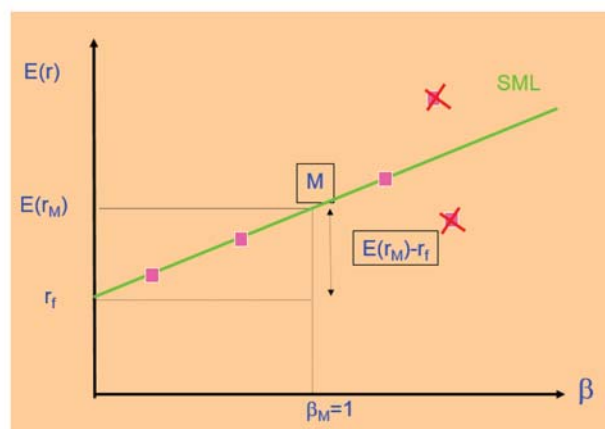
$$\beta_i \begin{matrix} > \\ = \\ < \end{matrix} 1$$

Aggressive
Average
Defensive

Interpretation Beta Coefficient

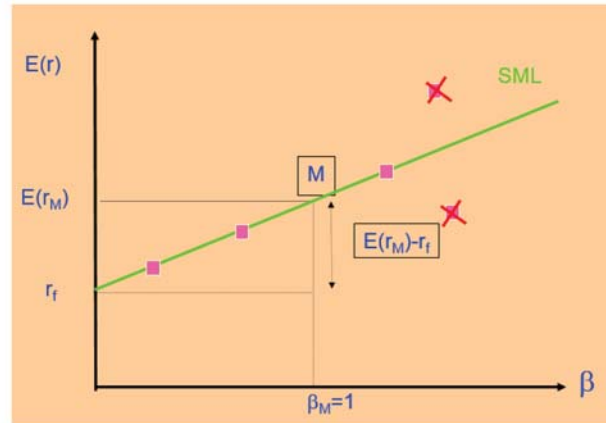
$$\begin{aligned}\beta_i &= \frac{\text{cov}(r_i, r_M)}{\sigma_M^2} = \frac{\text{cov}(r_i, r_M)}{\text{cov}(r_M, r_M)} \\ &= \frac{\text{cov}(r_i, r_M)}{\text{cov}(\sum w_i r_i, r_M)} \\ &= \frac{\text{cov}(r_i, r_M)}{\sum w_i \text{cov}(r_i, r_M)}\end{aligned}$$

The Security Market Line

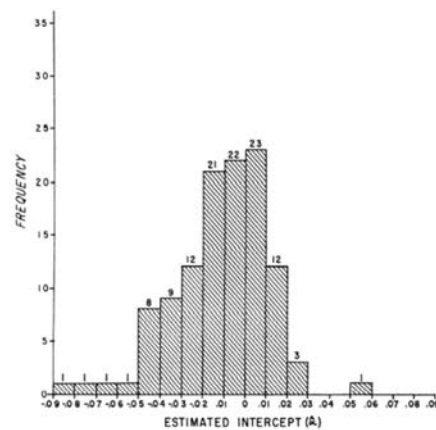


Alpha and Active Management

Expected versus required return



So Mutual Funds ?



Single Index Model and CAPM

- CAPM : expected returns
- SIM : realized returns
- CAPM : market portfolio
- SIM : a stock market index

$$R_i = \alpha_i + \beta_i R_M + e_i$$

$$\text{cov}(R_i, R_M) = \beta_i \sigma_M^2$$

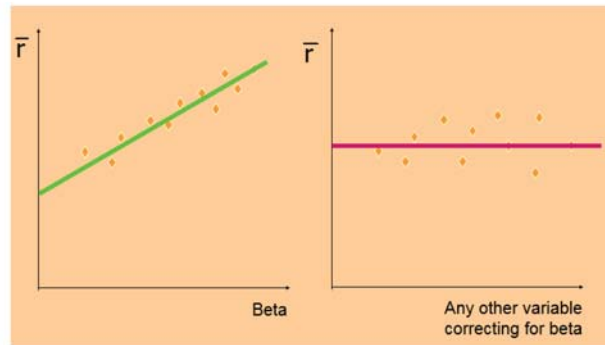
$$\beta_i = \frac{\text{cov}(R_i, R_M)}{\sigma_M^2}$$

Empirical Tests

$$E(r_i - r_f) = \beta_i E(r_M - r_f)$$

- Relation $E(r)$ and β is positive...
- ... and linear
- ... and complete

What Do We Expect?



First Step

Gather data

- returns r_{it} , $i = 1, 2, \dots, N$ and $t = 1, 2, \dots, T$
- market returns r_{mt}
- risk-free rates r_{ft}

Second Step

Compute Betas: ('first-pass regression')

- Estimate SCL for each asset i
- Keep this output:
 - b_i (OLS estimate β_i)
 - average $r_i - r_f$ (estimate expected excess return)
 - average $r_m - r_f$
 - residual risk i (as example 'any other variable')

Third Step

$$E(r_i - r_f) = \beta_i E(r_M - r_f)$$

Estimate SML ('second-pass regression')

$$\overline{r_i - r_f} = \gamma_0 + \gamma_1 b_i$$

- $\gamma_0 = 0$
- $\gamma_1 = \overline{r_m - r_f}$

Alternative Third Step

$$E(r_i - r_f) = \beta_i E(r_M - r_f)$$

$$\overline{r_i - r_f} = \gamma_0 + \gamma_1 b_i + \gamma_2 \sigma_{ei}^2$$

$$\gamma_0 = 0$$

$$\gamma_1 = \overline{r_m - r_f}$$

$$\gamma_2 = 0$$

Empirical Results

- SML is 'too flat'
- Estimated r_f is too high
- But
 - returns are volatile. . .
 - market portfolio (Roll!)
 - estimation errors in beta
 - can investors borrow at r_f ?

Estimation Errors and Beta

When beta is an estimate:

- slope too small
- intercept too high

Use portfolios instead of individual assets

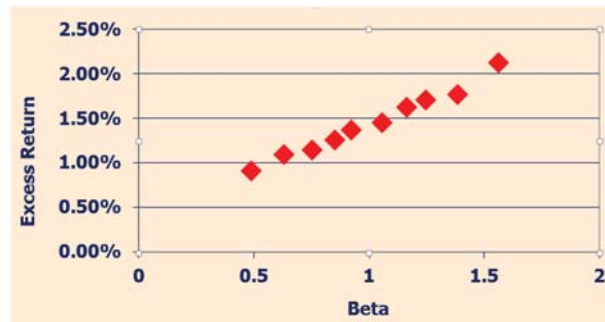
Less observations: therefore seek to obtain larger dispersion in beta

Rank on beta

Black, Jensen & Scholes

- Compute individual betas (5 year)
- Rank into deciles (6th year)
- Move forward by one year
- Repeat till end of sample

Results



Anomalies Reign...

Over the course of the years, the following “anomalies” have been documented:

- Size
- Leverage
- E/P
- B/M ...

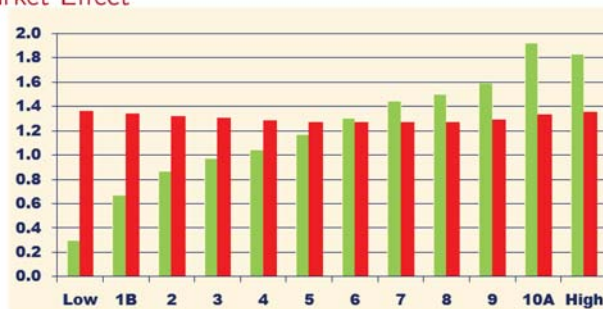
The Fama-French Assault

	Stock A	Stock B	Portfolio
Price t=0	100	50	150
Price t=1 (OK)	140	50	190
Price t=1 (Earthquake)	90	80	170
Expected Price	115	65	180
Expected return	15%	30%	20%

Source: Fama, E.F., & K.R. French, 1992, *The cross-section of expected stock returns*, *Journal of Finance* 47(2), 427-65.

The Fama-French Assault

Book-to-Market Effect



Source Fama and French (1992), Table IV

And There is More ...

- **Size** is negatively related to average return (Fama-French)
- **Reversals** e.g. Winner-Loser effect (De Bondt and Thaler (1985))
- **Momentum** (e.g. Jegadeesh & Titman (1993))

Is CAPM Dead?

- Extensions, e.g. Jagannathan and Wang (1996)
 - Market portfolio can be extended
 - Is beta constant?
- See later: multi-factor models
- But we need some additional theory: arbitrage pricing theory.

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Learning Objectives

- Understand what derivatives are
- Describe the contract specifications
- Describe the payoff diagrams
- Be able to read Futures quotes

Derivatives

- A **derivative** is an instrument whose payoff (and hence its value) depends on a characteristic of an underlying.
- The **underlying** can be anything: a financial asset (stock, bond), an index, the temperature, an event happening, ...
- Basic **derivatives contracts** are: Forwards, Futures, Swaps and Options
- There are always two **positions**: Long position (purchase of derivative) and Short position (sale of a derivative)
- Two main **motives** to trade: If you open a position, you speculate. If you close a position, you hedge.

Forward Contracts



On September 3, 2014, you went to FNAC to buy a copy of **Personal** by **Lee Child**, when it was just released.

Forward Contracts

- If the book was already in stock!
 - You paid €15.99 to purchase the book
 - Cash transaction
 - The deal was closed!
- If the book was already SOLD OUT
 - A demand for backorder @ €15.99 The book would be ready in 7 days
In 7 days you paid for and collected your copy.
 - This is a forward contract
 - A transaction for the delivery of an asset at a future point in time at a price agreed up on today.

Forward Contracts

September 3, 2014	September 10, 2014	Position
<u>You agree to</u> pay the purchase price of €15.99 and to receive the book when it arrives	<u>You</u> pay the purchase price of €15.99 to receive the book	<u>You</u> Long
<u>FNAC agrees to</u> accept payment of €15.99 and give up the book when it arrives	<u>FNAC</u> gives up the book (makes delivery) and accepts your payment of €15.99	<u>FNAC</u> Short

Forward Contracts

Definition

a binding agreement (obligation) to buy/sell an underlying asset in the future, at a price set today

A forward contract specifies

- The features and quantity of the asset to be delivered
- The delivery logistics, such as time, date, and place
- The price the buyer will pay at the time of delivery



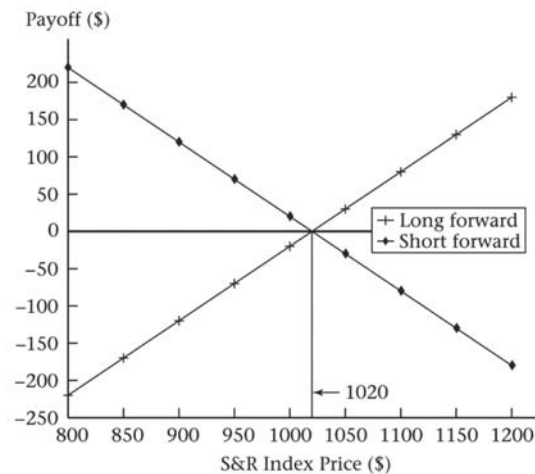
Forward Contracts

- Neither party can legally get out of the commitment.
- But they can offset their positions with someone else.
 - Based on the reviews of Lee Child, you decided not to buy this book.
 - You came to your neighbor, a big fan of Jack Reacher who happened to forget to order Lee Child and had to wait for two weeks.
 - You offered to sell the book to your neighbor to offset your long position: now you took a short position to sell the book.
 - You might even negotiate a better price!

Payoff of a Forward Contract

- Payoff of a contract is its value at expiration
 - Long forward = Spot price at expiration – forward price
 - Short forward = Forward price – spot price at expiration
- E.g. for non-dividend paying S&R (special & rich) index:
 - Today: Spot price = \$1,000, 6-month forward price = \$1,020
 - In six months at contract expiration: Spot price = \$1,050
 - Long position payoff = $\$1,050 - \$1,020 = \$30$
 - Short position payoff = $\$1,020 - \$1,050 = (\$30)$

Payoff Diagram of a Forward Contract



Forward Contracts: Example

A US company knows it will have to pay 3 million in euro in three months. The current exchange rate is 1.45 dollars per euro. Discuss how forward contracts can be used by the company to hedge its exposure.

Forward Contracts

Trade in over-the-counter (OTC) markets where traders working for banks, fund managers and corporate treasurers contact each other directly

- Not standardized
- Usually one specified delivery date
- Settled at the end of contract
- Some credit risk

Forward Contracts

Forward contracts are subject to a degree of uncertainty

- What if just one day after you asked FNAC to order one copy for you and suddenly you could buy this book in another book store at €5.99?
- What if the price in one week became €39.99 because it was extremely popular so FNAC did not want to sell it to you at the agreed price of €15.99 on September 10, 2014?
- Big forward contracts in real life faces even more risk.

Futures Contracts

The underlying

- Commodity: variation in the quality of what is available in the marketplace. Price received may depend on the grade chosen.
 - E.g. in the Chicago Mercantile Exchange Group's corn futures contract, the standard grade is "No. 2 Yellow". No. 1 Yellow is deliverable for 1.5 cents per bushel more than No. 2 Yellow.
- Financial assets: less ambiguous
 - E.g. Treasury bond futures on Chicago Board of Trade: the underlying bond can have a maturity between 15 and 25 years.

Futures Contracts

The contract size: The amount of the asset to be delivered under one contract

- Too small: transaction cost would be too high
- Too large: difficult to hedge small exposures
- E.g. a futures contract on an agricultural product might be \$ 10 000 to \$ 20 000. But one Treasury bond futures contract of Chicago Mercantile Exchange is for the delivery of bonds with a face value of \$ 100 000

Futures Contracts

Delivery arrangements

- The place where delivery will be made (specified by the exchange)
- E.g. IntercontinentalExchange (ICE) specifies the delivery of frozen concentrate orange juice contract: delivery is to exchange-licensed warehouses in Florida, New Jersey, or Delaware.
- The price received by the party with the short position is sometimes adjusted if alternative delivery locations are specified.

Futures Contracts

Delivery months

- The exchange specifies the precise PERIOD during the month when delivery can be made.
- E.g. corn futures traded by the CME Group have delivery months of March, May, July, September, and December.
- The broker of the party with the short position issues a notice of intention to delivery to the exchange clearing house.

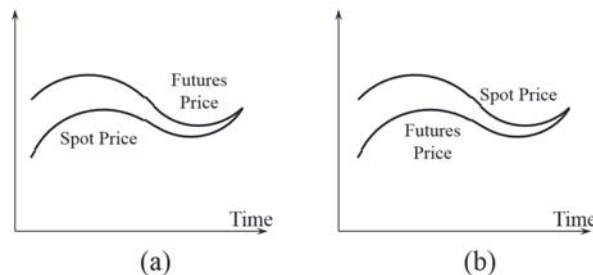
Futures Contracts

Price Quotes

- The exchange defines how prices will be quoted.
- E.g. in the US crude oil futures contracts, prices are quoted in dollars and cents.
- E.g. Treasury bond/note futures prices are quoted in dollars and thirty-seconds of a dollar: 134-16 represents $134 \frac{16}{32}$.

Convergence of the Futures Price to the Spot Price

- As the delivery period for a futures contract is approached:



- What if the futures price is above the spot price during the delivery period?

Futures Contracts

Margin account and daily settlement

- When you enter into a futures contract, the broker will ask you to deposit an amount of money when the contract is entered, known as the initial margin.
- At the end of each trading day, the margin account is adjusted to reflect the investor's gain or loss. This practice is called daily settlement or marking to market.
- The investor is entitled to withdraw any balance in the margin account in excess of the initial margin.
- If the balance in the margin account is lower than a maintenance margin, the investor will receive a margin call.

Futures Contracts

Margin accounts and daily settlement

- Suppose the current gold futures price is \$1,450 per ounce, with 100 ounces per contract. Suppose an investor has contracted to buy a total of 200 ounces at this price.
- We also suppose the initial margin is \$6,000 per contract and the maintenance margin is \$4,500 per contract.

Futures Contracts

Table 2.1 Operation of margin account for a long position in two gold futures contracts. The initial margin is \$6,000 per contract, or \$12,000 in total; the maintenance margin is \$4,500 per contract, or \$9,000 in total. The contract is entered into on Day 1 at \$1,450 and closed out on Day 16 at \$1,426.90.

Day	Trade price (\$)	Settlement price (\$)	Daily gain (\$)	Cumulative gain (\$)	Margin account balance (\$)	Margin call (\$)
1	1,450.00				12,000	
1		1,441.00	-1,800	-1,800	10,200	
2		1,438.30	-540	-2,340	9,660	
3		1,444.60	1,260	-1,080	10,920	
4		1,441.30	-660	-1,740	10,260	
5		1,440.10	-240	-1,980	10,020	
6		1,436.20	-780	-2,760	9,240	
7		1,429.90	-1,260	-4,020	7,980	4,020
8		1,430.80	180	-3,840	12,180	
9		1,425.40	-1,080	-4,920	11,100	
10		1,428.10	540	-4,380	11,640	
11		1,411.00	-3,420	-7,800	8,220	3,780
12		1,411.00	0	-7,800	12,000	
13		1,414.30	660	-7,140	12,660	
14		1,416.10	360	-6,780	13,020	
15		1,423.00	1,380	-5,400	14,400	
16	1,426.90		780	-4,620	15,180	

Futures Contracts

Closing out the positions

- The majority of futures contracts do not lead to delivery
 - The short can deliver on any of several delivery days and can choose to deliver any of several related but slightly different assets.
 - The long must accept whatever the short offers.
 - So the long often will do better to purchase it in the spot market.
- They are closed out prior to the delivery period

Futures Contracts: Exercise

A company enters into a short futures contract to sell 5,000 bushels of wheat for 750 cents per bushel. The initial margin is \$3,000 and the maintenance margin is \$2,000. What price change would lead to a margin call? Under what circumstances could \$1,500 be withdrawn from the margin account

Futures Contracts: Exercise

A trader buys two July futures contracts on frozen orange juice. Each contract is for the delivery of 15,000 pounds. The current futures price is 160 cents per pound, the initial margin is \$6,000 per contract, and the maintenance margin is \$4,500 per contract. What price change would lead to a margin call? Under what circumstances could \$2,000 be withdrawn from the margin account?

Futures Contracts: Exercise

In the corn futures contract, the following delivery months are available: March, May, July, September, and December. State the contract that should be used for hedging when the expiration of the hedge is in June?

Futures Contracts: Exercise

A portfolio manager has maintained an actively managed portfolio with a beta of 0.2. During the last year the risk-free rate was 5% and equities performed very badly providing a return of -30% . The portfolio manager produced a return of -10% and claims that in the circumstances it was good. Discuss this claim.

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Learning Objectives

- Understand how plain vanilla call and put options work
- Describe the contract specifications
- Be able to read option quotes
- Describe the payoff and profit potential of puts and calls
- Construct the payoff and profit diagrams of portfolios containing options

Nature

For their owner

- **Financial assets** represent a financial claim on an issuing organization
- **Options**
 - represent a negotiable right to buy or sell a financial asset.
 - are truncated assets
 - have no voting rights, no privileges of ownership, and no interest or dividend income
 - are created by individual investors, not by the issuing organization

Nature

- The option buyer
 - has the right to buy or sell an underlying asset for a given period of time, at a price that was fixed at the time of the option contract in exchange for paying the seller a fee
 - can walk away from a bad option
- The option seller / maker / writer
 - has the obligation to buy or sell the underlying asset according to the terms of the option contract, for which the seller has been paid a certain amount of money
 - cannot walk away from a bad option

Advantages of Options

- The investor can make money when value of assets go up or down
- The buyer's exposure to risk is limited to fee paid to purchase the option
- Allow to use leverage (i.e. the ability to obtain a given equity position at a reduced capital investment, thereby magnifying total return)

Advantages of Options

Example of leverage

Assume the market price for a share of common stock is \$50. A call option to purchase 100 shares of the stock at a strike price of \$50 per share may be purchased for \$500. If the market price of the stock goes up to \$75 per share, the buyer will purchase 100 shares at the strike price from the seller/maker/writer and sell them at the higher market price. The buyer's profit will be \$2,000.

The buyer's total return using the call option will be: 400%

The buyer's total return directly owning the stock would be: 40%

Disadvantages of Options

- Options expire
- Investor does not receive any interest or dividend income
- Options are perceived to be complex
- The seller's exposure to risk is unlimited
- Options are risky because an investor has to be correct on two decisions to make money:
 - Which direction the price of the asset will move
 - When the price change will occur

Quotations for Listed Stock Options

Option/Strike	Exp.	Vol.	--Call--		--Put--	
			Last	Vol.	Last	
IBM	130	Oct	364	15¼	107	5¼
138¼	130	Jan	112	19½	420	9¼
138¼	135	Jul	2365	4¼	2431	13/16
138¼	135	Aug	1231	9¼	94	5½
138¼	140	Jul	1826	1¼	427	2¼
138¼	140	Aug	2193	6½	58	7½

Quotations for Listed Stock Options

This option has a strike price of \$135;

Option/Strike	Exp.	Vol.	--Call--		--Put--	
			Last	Vol.	Last	
IBM	130	Oct	364	15¼	107	5¼
138¼	130	Jan	112	19½	420	9¼
138¼	135	Jul	2365	4¼	2431	13/16
138¼	135	Aug	1231	9¼	94	5½
138¼	140	Jul	1826	1¼	427	2¼
138¼	140	Aug	2193	6½	58	7½

a recent price for the stock is \$138.25

July is the expiration month

Quotations for Listed Stock Options

This makes a call option with this exercise price in-the-money by $\$3.25 = \$138\frac{1}{4} - \$135$.

Option/Strike	Exp.	Vol.	--Call--		--Put--	
			Last	Vol.	Last	Vol.
IBM	130	Oct	364	15¼	107	5¼
138¼	130	Jan	112	19½	420	9¼
138¼	135	Jul	2365	4%	2431	13/16
138¼	135	Aug	1231	9¼	94	5½
138¼	140	Jul	1826	1¾	427	2¾
138¼	140	Aug	2193	6½	58	7½

Puts with this exercise price are out-of-the-money.

Quotations for Listed Stock Options

Option/Strike	Exp.	Vol.	--Call--		--Put--	
			Last	Vol.	Last	Vol.
IBM	130	Oct	364	15¼	107	5¼
138¼	130	Jan	112	19½	420	9¼
138¼	135	Jul	2365	4%	2431	13/16
138¼	135	Aug	1231	9¼	94	5½
138¼	140	Jul	1826	1¾	427	2¾
138¼	140	Aug	2193	6½	58	7½

On this day, 2,365 call options with this exercise price were traded.

Quotations for Listed Stock Options

The CALL option with a strike price of \$135 is trading for \$4.75.

Option/Strike	Exp.	Vol.	--Call--		--Put--	
			Vol.	Last	Vol.	Last
IBM	130	Oct	364	15¼	107	5¼
138¼	130	Jan	112	19½	420	9¼
138¼	135	Jul	2365	4¾	2431	13/16
138¼	135	Aug	1231	9¾	94	5½
138¼	140	Jul	1826	1¾	427	2¾
138¼	140	Aug	2193	6½	58	7½

Since the option is on 100 shares of stock, buying this option would cost \$475 plus commissions.

Quotations for Listed Stock Options

Option/Strike	Exp.	Vol.	--Call--		--Put--	
			Vol.	Last	Vol.	Last
IBM	130	Oct	364	15¼	107	5¼
138¼	130	Jan	112	19½	420	9¼
138¼	135	Jul	2365	4¾	2431	13/16
138¼	135	Aug	1231	9¾	94	5½
138¼	140	Jul	1826	1¾	427	2¾
138¼	140	Aug	2193	6½	58	7½

On this day, 2,431 put options with this exercise price were traded.

Quotations for Listed Stock Options

The PUT option with a strike price of \$135 is trading for \$.8125.

Option/Strike	Exp.	Vol.	--Call--		--Put--	
			Last	Vol.	Last	Vol.
IBM	130	Oct	364	15 1/4	107	5 1/4
138 1/4	130	Jan	112	19 1/2	420	9 1/4
138 1/4	135	Jul	2365	4 1/4	2431	13/16
138 1/4	135	Aug	1231	9 1/4	94	5 1/2
138 1/4	140	Jul	1826	1 1/4	427	2 1/4
138 1/4	140	Aug	2193	6 1/4	58	7 1/2

Since the option is on 100 shares of stock, buying this option would cost \$81.25 plus commissions.

Quotations for Listed Stock Options

OPTION/STRIKE	EXP	VOL.	--CALL--		--PUT--	
			Last	Vol.	Last	Vol.
Albia	40	Aug	5563	0.30	6120	0.55
39.85	40	Sep	5124	1.60	670	2.20
39.85	42.50	Aug	4265	0.05	115	2.80
39.85	42.50	Sep	13177	0.65	239	4
AndriSp	20	Sep	3166	0.45	—	—
Autobahn	17.50	Aug	10273	0.65	—	—
18.10	17.50	Sep	10270	1	33	0.40
BEA Sys	12.50	Aug	9639	0.35	2794	0.25
Broadcom	20	Aug	3157	1.50	915	0.05
21.26	20	Sep	5230	2.35	5418	0.90
21.26	22.50	Sep	3136	1.05	478	2.05
Chemdex	60	Sep	7368	15.90	—	—
72.82	65	Sep	24151	9.20	58	0.10
72.82	70	Aug	18679	4	60	0.05
72.82	70	Sep	4238	4.10	191	0.40
Cris	15	Oct	144	3	5184	0.20
17.79	15	Jan	275	1.50	5395	0.60
17.79	17.50	Aug	9214	0.30	4356	0.05
Citigroup	45	Aug	5564	0.30	1655	0.10
CocaCola	45	Aug	10094	0.15	866	0.20
45.01	50	Feb	8238	0.80	—	—
Comcast sp	32.50	Oct	4489	0.60	—	—
Diamond	90	Aug	5384	3.30	250	0.05
93.33	91	Aug	5816	2.30	1667	0.05
93.33	92	Aug	3934	1.35	1922	0.05
93.33	93	Aug	3255	0.50	1983	0.20
DellInc	30	Aug	2490	1.60	6128	0.10
32.34	32.50	Aug	26994	0.15	1679	1.20
32.34	32.50	Sep	7396	0.70	650	1.70
Document	20	Oct	5510	0.70	—	—
ETrade	7.50	Jan	10403	1.75	18	0.70
17.34	10	Oct	5583	0.35	13	1.70
17.34	10	Jan	5233	0.65	5	2.10
EMC	11	Aug	3183	0.25	892	0.10
Edinet	15	Aug	16612	2.50	—	—
17.32	15	Sep	7350	2.90	3531	0.35
17.32	17.50	Aug	4435	0.30	120	0.20
17.32	17.50	Sep	5426	1	181	1.10
ExxonMob	37.50	Oct	4174	0.80	420	1.35
FordM	10	Dec	8862	1.25	103	0.65
Gen El	27.50	Aug	5235	1.05	30	0.05
28.78	30	Sep	10949	0.30	1079	0.05

Gitman & Joehnk, 2005, Fundamentals of Investing, Pearson.

Key Provisions of Stock Options

- **Strike Price**
 - Stated price at which you can buy a security with a call or sell a security with a put
 - Conventional (OTC) options may have any strike price
 - Listed options have standardized prices with price increments determined by the price of the stock
- **Expiration Date**
 - Stated date when the option expires and becomes worthless if not exercised
 - Conventional (OTC) options may have any working day as expiration date
 - Listed options have standardized expiration dates

Expiration Date of Listed Options

- **Three Expiration Cycles**
 - The January/April/July/October cycle
 - The February/May/August/November cycle
 - The March/June/September/December cycle
- The longest-term expiration dates are normally no longer than nine months
- The options that are longer than nine months are called LEAPS, and they are only available on some of the stocks
- Listed options always expire on the third Friday of the month of expiration

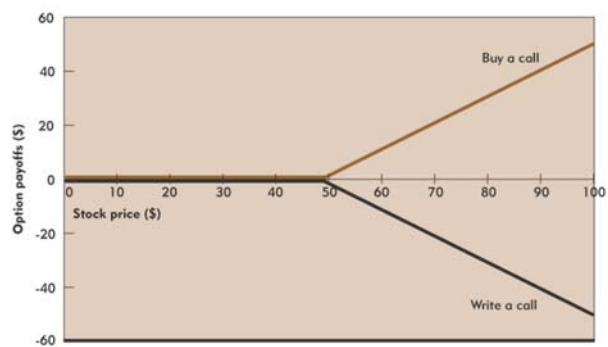
Traders Jargon

- In-the-Money
 - Call option: when the strike price is less than the market price of the underlying security
 - Put option: when the strike price is greater than the market price of the underlying security
- Out-of-the-Money
 - Call option: when the strike price is greater than the market price of the underlying security
 - Put option: when the strike price is less than the market price of the underlying security

Trading

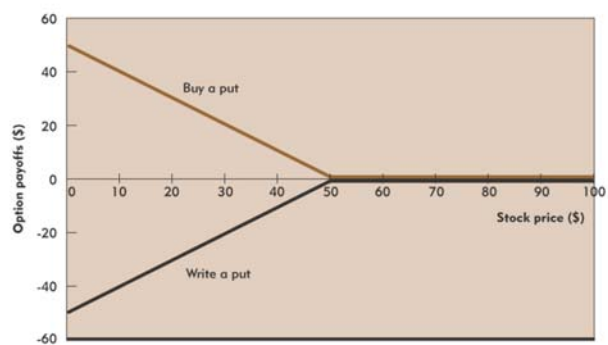
- Market Makers
 - Offer bid and ask quotes on the option.
 - Market makers ensure that buy and sell orders can always be executed at some price without delay.
 - The bid-ask spread is their compensation for this liquidity intermediation
- Offsetting Orders
 - An investor can close out a long position by issuing an offsetting order to sell the same option.
 - An investor can close out a short position by issuing an offsetting order to buy the same option.

The Call



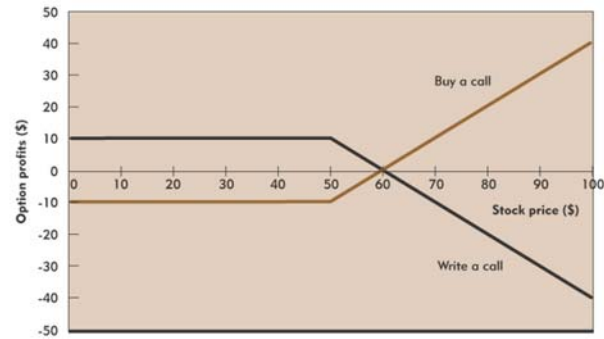
Source: Corrado and Jordan, Fundamentals of Investments, 3rd ed.

The Put



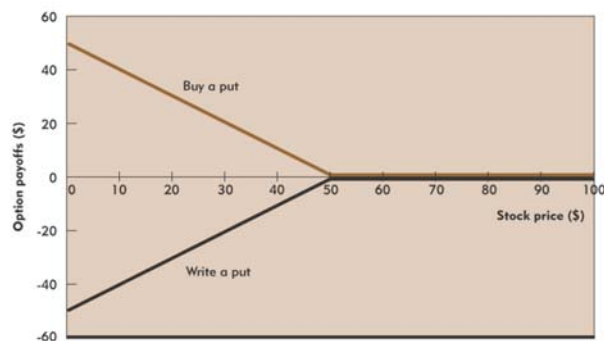
Source: Corrado and Jordan, Fundamentals of Investments, 3rd ed.

Calls



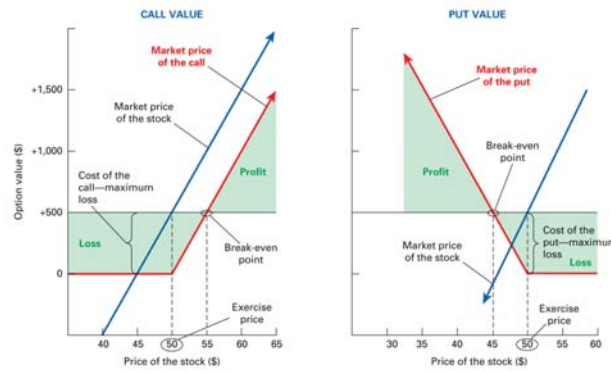
Source: Corrado and Jordan, Fundamentals of Investments, 3rd ed.

Puts

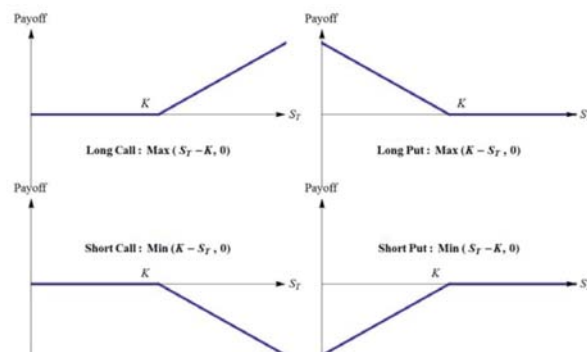


Source: Corrado and Jordan, Fundamentals of Investments, 3rd ed.

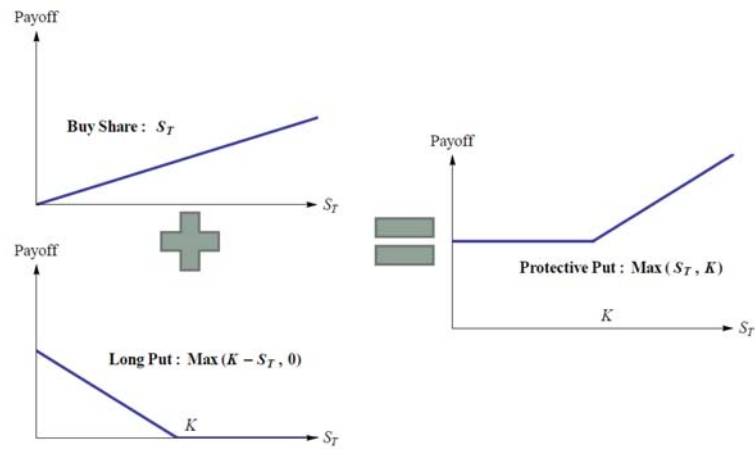
Profit Diagram



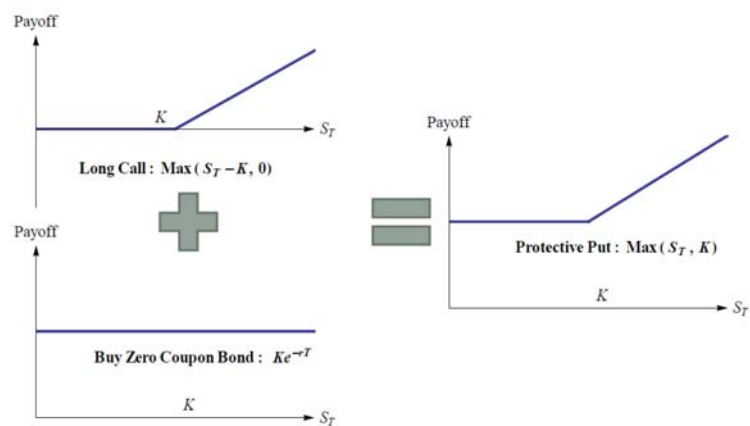
Option "Diamond"



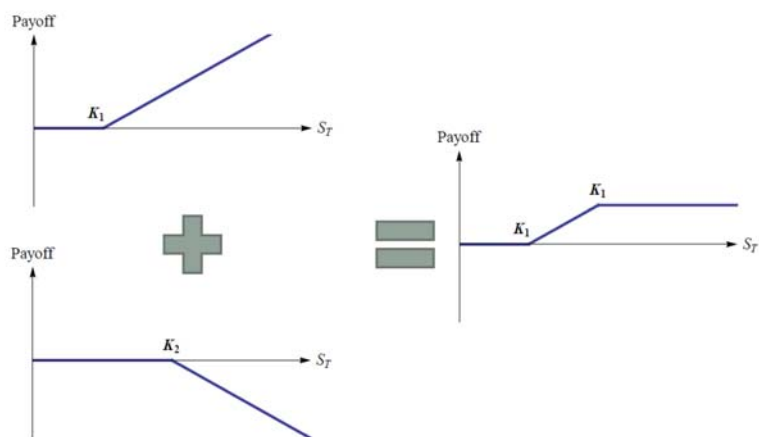
Protective Put



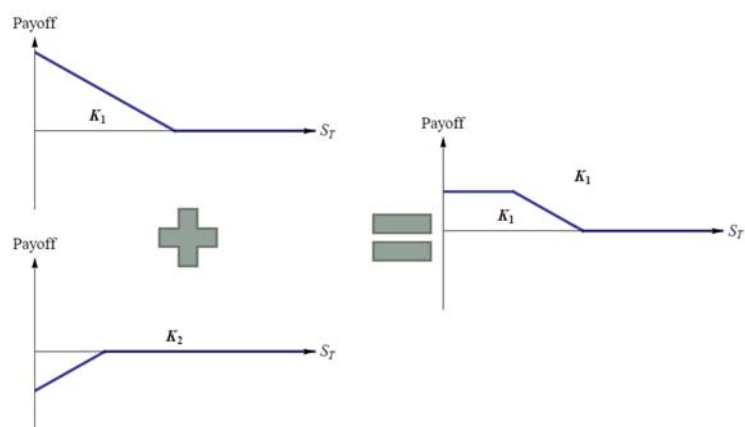
Fiduciary Call



Bull Spread



Bear Spread



The Synthetic Forward

